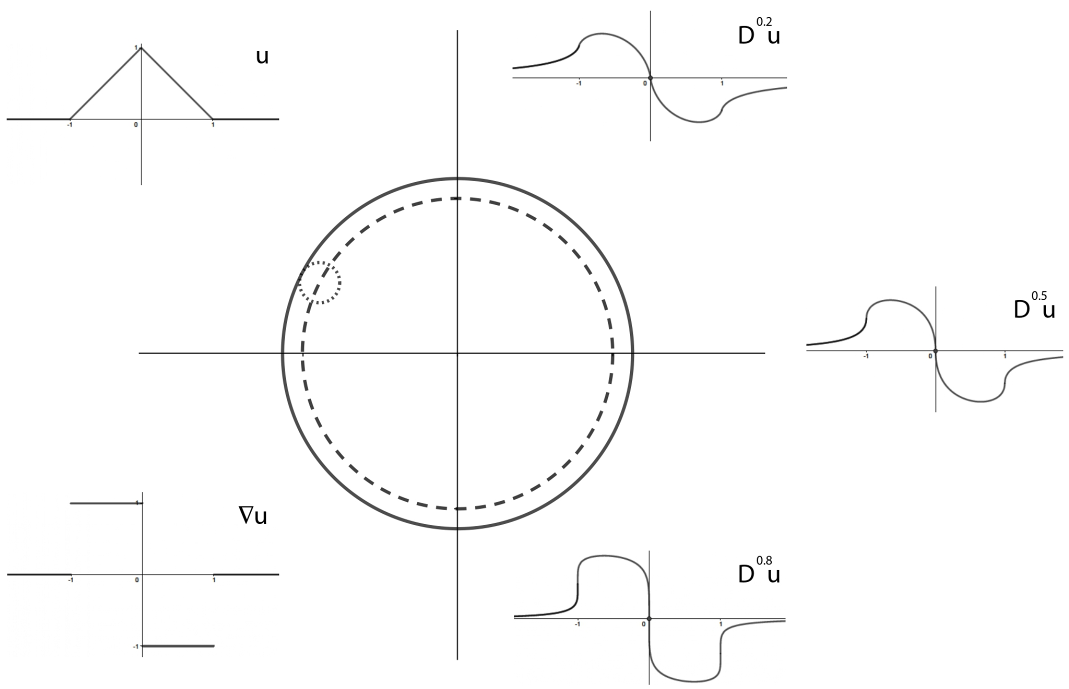
Abstract: This thesis is devoted to the study (through Calculus of Variations) of nonlocal models obtained from nonlinear Solid Mechanics that remains valid for discontinuous deformations. First, we address the suitability of bond-based peridynamics into nonlinear Solid Mechanics. The fact that not many hyperelastic functions can be recovered after the localization process leads us to study models and functionals whose energy density depends, in turn, on fractional (integral) gradients. This comes with a proper study of functional spaces based on the fractional gradient, fractional vector calculus and a recovering of the classical model when the fractional index s goes to 1. Finally, a third framework similar to the fractional one but acting over bounded domains is shown (relevant in applications). In this framework more tools had to be developed, including a nonlocal version of the fundamental theorem of Calculus. Finally, we manage to determine the existence of minimizers of nonlocal vector polyconvex energy functionals under (nonlocal) Dirichlet conditions in a functional space admiting functions exhibiting some singularity phenomena.



Mathematical analysis of fractional and nonlocal models from nonlinear Solid Mechanics

PhD thesis by

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Mathematical analysis of fractional and nonlocal models from nonlinear Solid Mechanics

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Summary

This thesis consists of three main parts devoted to the study of nonlocal models obtained from nonlinear Solid Mechanics. The motion of a system is often described by a set of integral or partial differential equations (PDE) so it is interesting to know if there are actual solutions for it. This is sometimes done (through Calculus of Variations) studying the existence of minimizers of the associated energy functionals (Section 1.2). In the first part of this thesis, we analyse the suitability of nonlinear bond-based peridynamics into Solid Mechanics, for which some restrictions are already known such as the fact that it requires the Poisson ratio to be $\nu = \frac{1}{4}$. The energy functional in this description can be written as a double integral on a pairwise potential function. The existence of minimizers (and thus, optimal deformations) was already shown in [23] as well as the localization process in order to recover classical hyperelastic energy functions [25]. Nevertheless, when we followed the study carried out in Part I, we discovered that not many hyperelastic functions can be recovered after the localization process. In particular, in Part I we recall such localization process depicted in [25] and follow it assuming certain physical conditions such as isotropy and frame-indifference, which are shown to remain after passing from the nonlocal density to the local one. Then, we obtain a condition for a classical hyperelastic function to be recovered from a (nonlocal) pair-wise potential function. This shows us that even in this simplified case, very few hyperelastic functions are recoverable. In particular, Money-Rivlin models cannot be recovered after localizing bond-based peridynamics. We also obtain the restriction of the Poisson ratio $\nu = \frac{1}{4}$, but through a different path. All this leads us to consider different kinds of nonlocal models, in particular those based on nonlocal operators such as the fractional gradient, or a nonlocal version over bounded domains. This means that instead of having a double integral as an energy functional, this would consist of an integral in terms of a function W which depends, in turn, of another integral operator. In other words, there would be a function between the two integrals imposing some difficulties at the time of working with these

fractional functionals or searching for bounds of the respective semi-norms.

Secondly, Part II is devoted to the study of (Bessel) fractional spaces, (a natural type of fractional spaces as shown in [99]) based on the Riesz s-fractional gradient, where the main goal is the study of vector fractional problems through Calculus of Variations. So as to do so, Part II is divided into three chapters. The first one focuses on the introduction and study of fractional (gradient) operators and their associated functional spaces according to some previous results in the literature. We highlight in particular [99, 100] where several important results such as continuous and compact embedding were already settled. However, a proper density result was not considered therein, although on the other hand, [37] tackled such issue from a distributional perspective. Such density issue is also addressed in this document but through a different approach. This chapter comes with some illustrative formulas that might help to provide a better insight with respect to these operators, showing in particular some analogous facts to the classical case, as a fractional integration by parts, the Fourier transform of the fractional gradient or its relationship with the, rather well-known, fractional laplacian. Nevertheless, not all of those formulas enjoy such similarity with the local case, as is the case of the fractional Leibniz rule, what will have some implications in the proof of the fractional Piola identity. This chapter ends with the aforementioned embeddings (with an alternative proof of the compact embedding) and two examples of functions in these fractional spaces exhibiting some singularity phenomena. In particular, in this chapter we have collected several results and continued developing the theory so that we had the tools to prove in the following chapter the existence of solution (a minimizer) of polyconvex vector fractional energy functionals under a complementary value condition in these Bessel spaces. This is done following a similar path to that of classical hyperelasticity [10]. In other words, assuming polyconvexity, it is required to prove the weak continuity of the determinant of the fractional gradient and then conclude using the direct method of Calculus of Variations. A key ingredient in this process is the fractional Piola identity, since it allows us to obtain a sort of integration by parts formula of the determinant. This chapter is completed with the corresponding Euler-Lagrange equations. Lastly, the third chapter is devoted to the study of the Γ-convergence (localization) when s goes to 1, for which some continuous inequalities in the first chapter (of this part) had to be written with independence of the index s. This, together with the convergence of the s-fractional gradients to the local one, leads to the recovering of the classical energy model over the whole space. A wider description is shown at the introduction of the chapters of Part II. Nevertheless, besides the clear academic interest of this approach, it may not be the most suitable one when one wants to deal with applications, since it is required the information from the whole space $\mathbb{R}^n$ to compute the fractional gradient as well as the respective energy functionals.

Finally, in Part III we consider a model which intends to fix the aforementioned drawbacks of the previous two parts. To wit, we study a nonlocal framework similar to that of Part II, i.e. based on a nonlocal gradient, similar to the Riesz s-fractional one but defined over bounded domains, giving rise to an energy functional whose energy density also depends on a fractional operator, having again a "function" between the two integrals of the energy functional. Given this structure, it seems that, as in the previous case, with this approach we could also recover the hyperelastic functions after the localization process (without the requirement of being defined over the whole space). All these facts would make this a proper model for nonlocal nonlinear Solid Mechanics, and in particular, as was suggested in [81], this kind of operators would fit in the state-based description of peridynamics. We would like to highlight the statement and proof of a nonlocal version of the fundamental theorem of calculus (to wit, an integral formula with which a function can be recovered from its nonlocal gradient), since it allows us to prove continuous embeddings as well as a sort of nonlocal mean value theorem, which allow us, in turn, to obtain a compact embedding result which were lacking in a nonlocal vector framework based on gradient operators. In particular, this nonlocal version of the fundamental theorem of calculus is rather involved and requires the use of advanced techniques from Fourier analysis. Since the localization process in this framework is not included in this manuscript, this last part consists only of two chapters. The structure is quite similar to that of Part II. The first chapter introduces the nonlocal operators and functional spaces we are going to deal with, in this case with a nonlocal (volumetric) boundary. We also show some formulas, results and embeddings in an analogous way to Part II except for the proof of a nonlocal version of the fundamental theorem of calculus, which required its own solo section. Then, as in the case with the fractional gradient, in the second and last chapter we address the existence of minimizers of nonlocal vector energy functionals under convexity as well as polyconvexity, for which it is also required a nonlocal version of the Piola identity. Accordingly, the Euler-Lagrange equations are also obtained.

Resumen

Esta tesis consta de tres partes diferenciadas, con el fin de estudiar modelos no locales obtenidos a partir de la Mecánica de Sólidos no lineal. El comportamiento de un sistema es a menudo descrito por un conjunto de ecuaciones en derivadas parciales (EDPs) o ecuaciones integrales, por lo que es interesante ver si realmente hay alguna solución al sistema para describir dicho comportamiento. Ésto es algo que en algunos casos se hace (a través del Cálculo de Variaciones) estudiando la existencia de minimizadores del funcional energía asociado (Sección 1.2). En la primera parte de la tesis analizamos la idoneidad de la descripción *bond-based* de la peridinámica en la mecánica de sólidos, para la cual ya se conocía alguna restricción, como el hecho de que impone que el ratio de Poisson sea $\nu = \frac{1}{4}$. El funcional energía acorde con esta formulación viene dado por una doble integral de una función densidad sobre cada par de puntos. La existencia de minimizadores (y por tanto, de deformaciones óptimas) ya fue probada en [23] así como el proceso de localización para recuperar funciones clásicas de hiperelasticidad cuando la no localidad desaparece [25]. Sin embargo, al desarrollar el estudio mostrado en la Parte I, observamos que no muchas funciones hiperelásticas se pueden recuperar después del proceso de localización. En particular, se recuerda dicho proceso de localización mostrado en [25] y éste es reproducido para el caso más sencillo en el que se asumen ciertas condiciones físicas naturales como son la isotropía y la invariancia con respecto al observador. También se muestra que dichas propiedades se siguen cumpliendo al pasar de la densidad no local a la local. A continuación, se obtiene una condición para caracterizar las funciones clásicas de hiperelasticidad que pueden ser recuperadas a partir de una función de densidad evaluada sobre cada par de puntos. Ésto nos muestra que incluso en este caso más sencillo, hay muy pocas funciones hiperelásticas que pueden ser recuperadas. Adicionalmente, también se obtiene en este documento la restricción sobre el ratio de Poisson $\nu = \frac{1}{4}$, pero por medio de un camino diferente. Todo ésto nos lleva a considerar otros tipos diferentes de modelos no locales, en particular, a aquellos que están basados directamente

en operadores no locales como el gradiente fraccionario, o una versión suya no local que opere sobre dominios acotados. Ésto implica que en vez de tener una doble integral como funcional de energía, ésta consistiría de una integral sobre una función W, que depende, a su vez, de otro operador integral. En otras palabras, habría una función entre las dos integrales, lo cual impondría algunas dificultades a la hora de trabajar con estos funcionales fraccionarios o en la búsqueda de cotas adecuadas de las respectivas semi-normas.

La Parte II de la memoria está dedicada al estudio de los espacios fraccionarios (de Bessel), (una especie natural de espacios fraccionarios como se mostró en [99]) basados en el gradiente fraccionario s de Riesz, donde el principal objetivo es el estudio de problemas vectoriales fraccionarios a través del Cálculo de Variaciones. Para ello, la Parte II se divide en tres capítulos. El primero de ellos se centra en la introducción y estudio de operadores (gradiente) fraccionarios y sus espacios funcionales asociados de acuerdo con algunos resultados previos en la literatura. Resaltamos en concreto [99,100], donde varios resultados importantes como son las inmersiones compactas y continuas ya fueron establecidos. Sin embargo, en dichos artículos no se consideró un resultado de densidad en sí, aunque por otra parte, en [37] abordaron dicho tema desde una perspectiva distribucional. Dicho asunto de la densidad también se ha abordado en este documento, pero por medio de un enfoque diferente. Este capítulo incluye algunas fórmulas ilustrativas que pueden ayudar a proporcionar una mejor percepción de estos operadores, como por ejemplo, mostrando algunos hechos análogos a los que se dan en el caso clásico, como son una integración por partes fraccionaria, la transformada de Fourier del gradiente fraccionario o la fórmula que lo relaciona con el más conocido laplaciano fraccionario. Sin embargo, no todas esas fórmulas consiguen tal similaridad con el caso local, como es el caso de la derivada fraccionaria del producto, lo cuál tendrá sus implicaciones en la prueba de la identidad de Piola fraccionaria. Este capítulo acaba con las previamente mencionadas inmersiones (incluyendo una prueba alternativa de la inmersión compacta) y dos ejemplos de funciones en estos espacios fraccionarios que representan singularidades, como las fracturas o cavitaciones. En particular, en este capítulo se han recogido varios resultados y se ha continuado desarrollando la teoría de forma que se tengan las herramientas necesarias para probar en el siguiente capítulo la existencia de solución (de minimizadores), en estos espacios de Bessel, de funcionales de energía vectoriales, fraccionarios y policonvexos bajo una condición en el complementario del dominio (que sustituye a la condición de frontera). Ésto se ha llevado a cabo siguiendo un camino similar al tomado en el caso de la hiperelasticidad clásica [10]. En otras palabras, asumiendo la condición de policonvexidad, los pasos son probar la continuidad débil del determinante del gradiente fraccionario y luego concluir usando el método directo del Cálculo de Variaciones. Un ingrediente clave en todo este proceso es la identidad de Piola fraccionaria, dado que ésta nos permite obtener una especie de fórmula de integración por partes del determinante. Para completar este capítulo se incluyen las correspondientes ecuaciones de Euler-Lagrange. Por último, el tercer capítulo de esta parte está dedicado al estudio de la Γ-convergencia (localización) cuando s tiende a 1, para el cuál algunas desigualdades en el primer capítulo de esta parte han tenido que ser escritas con una constante independiente de s. Ésto, junto con el hecho de que el gradiente fraccionario converge al gradiente clásico cuando s tiende a 1, nos lleva a recuperar el modelo clásico (definido sobre todo el espacio). Sin embargo, a pesar del claro interés académico de este enfoque, puede que no sea el más adecuado cuando uno quiere lidiar con aplicaciones, dado que se requiere la información sobre todo el espacio $\mathbb{R}^n$ para obtener el gradiente fraccionario así como los respectivos funcionales de energía.

Finalmente, en la Parte III consideramos un modelo que busca suplir los mencionados inconvenientes de las dos Partes anteriores. Es decir, se estudia un marco no local similar al de la Parte II, que está basado en un gradiente no local similar al gradiente fraccionario, pero definido sobre dominios acotados de forma que llegamos a tratar de nuevo con funcionales de energía definidos como una integral cuya función densidad de energía también depende a su vez de otra integral, teniendo de nuevo una función entre dos integrales del funcional energía. Dada esta estructura, parece que, como en el caso anterior, con este enfoque se podrían recuperar también las funciones hiperelásticas después del proceso de localización (sin necesidad de que estén definidas sobre todo el espacio). Todas estas observaciones harían de éste un modelo adecuado para la Mecánica de Sólidos no lineal, no local, y en particular, como se comenta en [81], este tipo de operadores encajaría en la descripción *state-based* de la peridinámica. En esta parte, destaca sobre lo demás, como mayor novedad en su análisis, el enunciado y prueba de una versión no local del teorema fundamental del cálculo (es decir, una fórmula integral con la que recuperamos una función a partir de su gradiente no local), dado que éste nos permite probar inmersiones continuas así como una especie de teorema del valor medio no local, que nos permiten, a su vez, probar un resultado de inmersión compacta que faltaba en un marco no local vectorial basado en operadores gradiente. En concreto, la prueba de esta versión no local del teorema fundamental del cálculo involucra complejidad y requiere el uso de técnicas avanzadas de análisis de Fourier. Dado que el proceso de localización en este contexto no está incluido en esta memoria, esta última parte consta sólo de dos capítulos. La estructura es bastante similar a la de la Parte II. En el primer capítulo se introducen los operadores no locales y espacios funcionales con los que vamos a tratar, que en este caso conlleva lidiar también con una frontera no local (volumétrica). También se muestran algunas fórmulas, resultados e inclusiones de forma análoga a la Parte II excepto por la prueba de la mencionada versión no local del teorema fundamental del cálculo, el cuál requiere una sección propia. Luego, como en el caso del gradiente fraccionario, en el segundo y último capítulo abordamos la existencia de minimizadores de funcionales de energía vectoriales no locales bajo condiciones de convexidad, así como de, la más general, policonvexidad. De manera acorde también se obtienen las ecuaciones de Euler-Lagrange para este caso.

Contents

Chapter 1.
Introduction

Modelling nature has always been a point of interest for physicists, mathematicians and other scientists, who found in mathematics a relevant language and useful tools so as to describe such different processes. One of the most notable ways to do so is by means of (partial) differential equations which frequently appear in mathematics, physics, engineering and biology. These equations have become quite popular in approximating or describing different phenomena in several fields. However, they have a strong dependence on the classical gradient operator (derivative) which gives them their local nature, usually imposes some regularity and prevents them from catching some nonlocal information. In the last decades, nonlocal problems are gathering more attention and being considered as alternative or supplementary models to local ones, since they can catch some information that the previous ones cannot, such as long range interactions. Specifically, they may require less regularity of the functions, allowing for more general admissible solutions. In fact, typical characteristics of nonlocal problems include being able to provide an effective modelling for discontinuities or singularities and computing interactions through integration instead of differentiation (which gives rise to integral or integro-differential equations), which is related to the fact that points separated by a finite distance exert a force upon each other. This has led to the development of several studies considering their applications to various fields such as continuum mechanics [51, 52, 103, 107], image processing [9, 34, 42, 64], nonlocal diffusion [3, 35, 114] or machine learning [4, 87]. See also [47] for an introduction to nonlocal modelling. This interest in nonlocal problems has been accompanied by a budding but significant progress on its mathematical development.

A feature that frequently appears in several models, is the fact that the range of nonlocal interactions is determined by a positive value, which is generally given by the radius of an Euclidean ball as in the case of peridynamics theory, (where interactions between particles is assumed to be negligible when they are further away than a certain distance), which is one of the main motivations of this work. Peridynamics is a nonlocal alternative model in Solid Mechanics introduced by S.A. Silling in [103] (bond-based peridynamics), who later added the more general state-based description of peridynamics [107]. Peridynamics theory was proposed in order to overcome some modelling aspects since, although classical elasticity models (whose use is rather spread among the scientific and engineering communities) have proved to be quite practical in certain situations, they still fail when singularity phenomena such as fracture or cavitation appears. As with other nonlocal problems, the development of this theory in the last years, though recent, has been impressive. Some references on this are [72, 75, 104, 107] and the two books [62, 77].

All these advantages in modelling that nonlocal models offer may come at a cost. Concretely, this usually means that we have to deal with some complex (integral) terms, which pose some technical difficulties in the mathematical analysis and requires working with more general functional spaces. Another relevant aspect is that of boundary conditions which require a better understanding since they could be taken as of volumetric type given their nonlocal nature. Although a significant part of the development of nonlocal problems has been driven by a practical and applied interest, one of the most noticeable disadvantages of this problem comes at the time of numerical simulations since they often impose an increased computational cost, given the introduction of more complex, integral operators in the model. There have been several works dealing with this issue [43, 45, 60].

Delving a little bit into the structure of nonlocal models, it is usual to see them obtained by means of the integration against a nonlocal kernel which sometimes can define a nonlocal gradient operator,

$$\int \frac{u(x)-u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|}\, dy.$$

A particular case is that of fractional calculus, where thanks to a particular kernel the notion of derivative is extended to non-integer differentiability indexes $s \in (0,1)$ which gives rise to spaces of functions that may admit discontinuities. In one dimension there are a lot of different notions, where the Lioville and Caputo's derivative stand out . On the other hand, in several dimensions the so-called fractional laplacian has captured and enjoyed most of the attention and development in this framework whereas the Riesz fractional gradient, which is related to the former in a completely analogous way to their local counterparts, is starting to take off. Actually, [102] showed that the Riesz fractional gradient enjoys several characteristics such as uniqueness up to a multiplicative constant under natural requirements, making it the natural definition for a fractional differential object.

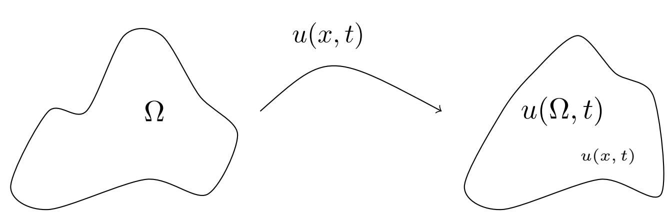
We believe that the results showed in this work would be of help so as to obtain a better understanding of the nature underlying these models as well as part of the mathematical structure and provide tools that may make the handling of nonlocal models more treatable.

Hence, the goal of this work is to obtain and study a nonlocal model of hyperelasticity (fully nonlinear) so that it remains valid for discontinuous deformations. So as to do so we first study the suitability of bond-based peridynamics when we are dealing with nonlinear problems in Solid Mechanics. Then we study and continue developing the mathematical foundation of fractional and nonlocal vector calculus that will allow us to cast the sought energy functionals as well as the existence of minimizer under polyconvexity.

Given the subjects of Solid Mechanics and Calculus of Variations to which this work belongs, we provide an introduction to nonlinear elasticity, vector variational problems as well as peridynamics, talking about the two mentioned descriptions, bond-based and stated-based ones.

1.1 Nonlinear elasticity

Classical elasticity theory is the branch of Solid Mechanics that deals with deformations that are reversible, i.e. when a material recovers its original shape after being deformed by an external load. In particular, nonlinear elasticity deals with deformations that can be either large or small, so there could be a significant difference between the *deformed configuration* of a material and its original one (*reference configuration*). It is usually denoted by $u(x,t) \in \mathbb{R}^3$ the position occupied by the material at the point x in a domain Ω and at the time t. Hence, u is the function that takes the reference configuration to the deformed configuration $u(\Omega, t)$.



Further, the deformation u must fulfil the physical requirements of preservation of the orientation and non-interpenetration of matter. The former translates mathematically into the requirement of u being injective, while the latter into the condition $\det Du(x) > 0$ a.e. $x \in \Omega$, where Du is the deformation gradient, which is itself a measure of strain. As it is usual, the equations of motion are obtained after imposing Newton's second law, i.e.

$$\rho_R \ddot{u} = \operatorname{div} T(x) + F(x),$$

where $\rho_R \ddot{u}$ stands for the density times the acceleration, the term F represents the external body force, while $\operatorname{div} T(x)$ stands for the interactions of inner particles and $T(x)$ is the Piola-Kirchhoff stress tensor, which is related to the Cauchy stress tensor $\mathcal{T}$ through the identity

$$T(x) = \det Du(x)\mathcal{T}(u(x))Du(x)^{-T} = \mathcal{T}(u(x))\operatorname{cof}(Du(x))^T,$$

where $\operatorname{cof} A$ refers to the cofactor matrix of A. This hyperbolic system was shown to have existence by T.J.R. Hughes, T.Kato and J.E. Marsden but under not totally realistic assumptions [70].

Focusing on the stationary model, classical theory of Solid Mechanics establishes that the deformation produced on the body by external loads must verify the following conditions written in the reference configuration

$$\begin{cases} -\operatorname{div}(T(x)) = F(x), & x \in \Omega, \\ T(x) \cdot n = g(x), & x \in \Gamma_1, \end{cases}$$

where Γ_1 is the subset of the boundary $\partial\Omega$ where the surface force g is applied, and n is the outer normal to Γ_1.

So as to obtain the constitutive equation, a material is mathematically defined as elastic if the Cauchy stress tensor $\mathcal{T}(y)$ in each point of a deformed configuration $y \in u(\Omega)$ is a function exclusively of $x = u^{-1}(y)$ and the deformation gradient $Du(x)$. In this case, the Piola-Kirchhoff stress tensor is now

$$T(x) = T(x, Du(x)).$$

Assuming that a Dirichlet boundary condition u_0 is imposed on $\Gamma_0 = \partial\Omega \backslash \Gamma_1$, the deformation u must satisfy the following boundary value problem

$$\begin{cases} -\operatorname{div}(T(x, Du(x))) = F(x), & x \in \Omega, \\ T(x, Du(x)) \cdot n = g(x), & x \in \Gamma_1, \\ u = u_0, & x \in \Gamma_0. \end{cases}$$

We assume here that (as is common in a majority of the interesting applications) body and boundary forces, respectively F and g, do not depend on the deformation u.

The existence of solution of this stationary problem (besides some contemporary results by T.Valent in [113]) with realistic assumption was solved by John Ball applying Calculus of Variations [10]. An important concept in such process is hyperelasticity. An elastic material is called hyperelastic if there exists a function $W : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$, called stored-energy function of the material, such that the Piola-Kirchhoff stress tensor is the derivative of W with respect to its second variable.

$$T(x, A) = \frac{\partial W}{\partial A}(x, A), \quad x \in \Omega, \quad A \in \mathbb{R}^{m\times n}.$$

In this case, the *potential elastic* energy of the deformation u is

$$E(u) = \int_\Omega W(x, Du(x))\, dx - \int_\Omega F(x) \cdot u(x)\, dx - \int_{\Gamma_1} g(x) \cdot u(x)\, d\mathcal{H}^{n-1}(x),$$

where $F : \Omega \to \mathbb{R}^m$ is the distributed body load and $g : \Gamma_1 \subset \partial\Omega \to \mathbb{R}^m$ the boundary load. The term $d\mathcal{H}^{n-1}$ indicates the Lebesgue $(n-1)$-surface integral. Moreover, critical points of the energy are equilibrium solutions of Cauchy equations of motion.

For consistency from the physical point of view, it is natural to impose some conditions as the frame indifference. Thus, it is considered that the stored-energy density must satisfy the frame indifference condition

$$W(x, RA) = W(x, A) \tag{1.1}$$

for all points $x \in \Omega$, all matrices $A \in \mathbb{R}^{m\times n}$ and all rotations $R \in SO(m) = \{Q \in \mathbb{R}^{m\times m} \colon \det Q = 1,\, QQ^T = I_m\}$ (I_m the identity matrix of order m). This reflects the fact that the deformation energy does not depend on the observer.

Another physical property that we are going to consider throughout this document is concerned about symmetry of the stored energy density function. In particular, we introduce the definition for isotropic materials. A hyperelastic material is called isotropic if

$$W(x, AR) = W(x, A), \tag{1.2}$$

for all points $x \in \Omega$, all matrices $A \in \mathbb{R}^{m\times n}$ and all rotations $R \in SO(n)$. This means that the elastic energy is independent of the stretching, or loading, direction. When $n = m = 3$, Rivlin-Ericksen representation theorem (see [36, Theorem 4.3-1]) establishes that the hyperelastic body is isotropic if and only if, at each point, the stored-energy function depends only on $|A|^2$, $|\operatorname{cof} A|^2$ and $\det A$; in other words, if and only if there exists $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$W(x, A) = \varphi(x, |A|^2, |\operatorname{cof} A|^2, \det A).$$

A paradigmatic example of isotropic hyperelastic materials are Mooney-Rivlin materials, whose stored-energy density is given by the expression

$$W(A) = \alpha|A|^2 + \beta|\operatorname{cof} A|^2 + g(\det A), \tag{1.3}$$

with $\alpha, \beta > 0$ and g a given function. When $\beta = 0$, we recover the Neo-Hookean materials. Although the hyperlasticity assumption might be seen like another restriction, it seems that most materials can be modelled as hyperelastic ones: it was shown in [68, Section 28] that a necessary and sufficient condition for an elastic material to be hyperelastic is for "the work to be nonnegative in closed processes", and according to the beginning of [68, Section 28], nonnegative work in closed processes is a requirement of thermodynamics. Actually, recent works show some modelling of biological tissues, like [61] with an accurate model of the heart, or are even used in the film industry in order to make their animations more realistic [108].

An essential reference on the mathematical theory of nonlinear elasticity is [36]. Other references are the books [5, 39, 78, 89] and the survey papers [12, 14].

1.2 Vector variational problems

1.2.1 Introduction and historical perspective

Previously to the proper theory of Calculus of Variations some problems appeared describing a law in mathematics or nature by minimizing a certain magnitude, i.e. minimization principles. That is the case of the law of reflection (specular reflection) stated by Hero of Alexandria, who said that among every possible path between two points, the light take the shortest one. A more general principle than that of reflection was given by Fermat in order to include the refraction process through different media into the problem and using variational techniques. The principle says that a ray of light between two given points follows the path that minimizes time. Another problem whose statement dates from antiquity is the isoperimetric inequality. It can be posed as an optimization problem: either finding the curve of fixed length enclosing the maximal area, or the one with minimal length given a fixed area. This problem was later reformulated within Calculus of Variations terminology. Other problems arose such as finding the minimal path throughout not necessarily flat surfaces or the search of minimal surfaces with some constrains on their surface measure or boundary (Plateau's problem). This phenomena can be observed in the shapes adopted by soap films. A special mentions goes for the brachistochrone curve problem (i.e. finding the curve taken by a descending ball between two points that minimizes time), a hallmark in the history of Calculus of Variations which was proposed as a challenge among the Bernoulli brothers, L'Hôpital, Leibniz and Newton. A introductory text to Calculus of Variations with a historical perspective can be found in [112]. Another introductory book to Optimization and Calculus of Variations can be found in [90].

In general, there are many problems in nature where the system tends to have a state of minimal energy. This is reflected in the Principle of least action (or stationary action). It says that given the generalized coordinates $q(t)$ that show the evolution of the system, the Lagrangian $L[q(t), \dot{q}(t), t]$ and the action defined as

$$S[q(t)] = \int_{t_1}^{t_2} L[q(t), \dot{q}(t), t]\, dt$$

for times $t_1 < t_2$, then the trajectory (solutions of the equations of motion) between times t_1 and t_2 (and images $q_1 = q(t_1)$ and $q_2 = q(t_2)$) is a critical point of $S[q(t)]$ (i.e. zeros of the derivative).

Some years later, Euler and Lagrange showed that the minimizers of a given functional

$$F(y) = \int_{x_1}^{x_2} f(x, y(x), y'(x))\, dx$$

can be found among the solutions of the corresponding Euler-Lagrange equations.

$$\frac{\partial f}{\partial y}(x, y(x), y'(x)) - \frac{d}{dx}\frac{\partial f}{\partial y'}(x, y(x), y'(x)) = 0.$$

Equivalently, under certain conditions, the existence of solution of a system of equations of motion can be determined by proving the existence of a minimizer of their energy functional. This assertions can also be obtained in a framework of several variables as it is the case of hyperelasticity theory mentioned in the previous section.

1.2.2 Calculus of Variations

The Calculus of Variations is a branch of mathematical analysis that focuses on studying extremals and critical points of functionals that may depend on one or several functions (and their derivatives), which might be abided to several constrains of different nature. For relevant references on the Calculus of Variations see [8, 39, 65, 78, 89, 95]. This theory also provides an approach to determine the existence of solutions of certain sets of equations under some conditions. Concretely, for $I(u)$ an integral functional, we consider the problem

$$\min\{I(u)\,;\, u \in V\}.$$

An important tool in this framework is the **direct method of Calculus of Variations**. It is a way of determining the existence of solution (a minimizer) of a variational problem provided the following two ingredients.

1. **Coercivity:** $\lim_{\|u\|\to\infty} I(u) = +\infty$
2. **Sequential lower semi-continuity:** For every $u_j \rightharpoonup u$ (weakly), we have that
$$I(u) \leq \liminf_{j\to\infty} I(u_j)$$

with $u : \mathbb{R}^n \to \mathbb{R}^m$. The first ingredient is that the functional must be coercive, in the sense that $I(u)$ blows up as the norm of u increases (a compactness property). For the second ingredient, I must be lower semi-continuous; in our case, weak lower semicontinuous, since the relevant topology in this situation is the weak topology in a Sobolev space. Coercivity is usually guaranteed by imposing proper lower bounds on the stored-energy density, i.e. the integrand function. More delicate is the weak lower semicontinuity, which, for integral functionals like I, is typically characterized in terms of convexity notions for the integrand. In the scalar case ($n = 1$ or $m = 1$), the standard definition of convexity guarantees the weak lower semicontinuity [39].

In the pioneering work [10], an existence theory in hyperelasticity was given by means of the application of the direct method of the Calculus of Variations to the energy functional

$$I(u) = \int_\Omega W(x, Du(x))\, dx. \tag{1.4}$$

In Solid Mechanics we are usually concerned about vectorial problems ($n, m > 1$). In this case several options weaker than convexity appear which may also provide the weak lower semicontinuity of the functional. In this situation, the relevant convexity concept is quasiconvexity (see [39] and the references therein). We say that a function $\psi : \mathbb{R}^{m\times n} \to \mathbb{R}$ is *quasiconvex* if and only if

$$\psi(A) \leq \int_{(0,1)^n} \psi(A + Dv(x))\, dx, \tag{1.5}$$

for all matrices $A \in \mathbb{R}^{m\times n}$ and test functions $v \in C_c^\infty((0,1)^n, \mathbb{R}^m)$. It turns out that under the standard coercivity and growth conditions,

$$\frac{1}{C}|A|^p - C \leq W(x, A) \leq C(1 + |A|^p), \qquad \text{a.e. } x \in \Omega, \quad A \in \mathbb{R}^{m\times n},$$

for some $1 < p < \infty$ and $C > 0$, and assuming that $W(x, \cdot)$ is quasiconvex for a.e. $x \in \Omega$, the existence of minimizers for I holds. In hyperelasticity, quasiconvexity of isotropic hyperelastic stored-energy densities is a consequence of its polyconvexity. When $n = m = 3$, we say that W is **polyconvex** if it can be expressed as

$$W(x, A) = \varphi(x, A, \operatorname{cof} A, \det A)$$

for some φ such that $\varphi(x, \cdot, \cdot, \cdot)$ is convex for a.e. $x \in \Omega$ (see [39] for the definition of polyconvexity for general dimensions). For a general definition see Part II Section 4.3. Polyconvexity implies quasiconvexity under proper coercivity and growth conditions [10, 39], and, therefore, it is the right convexity notion in this context. When dealing with polyconvex densities, upper bounds can be left behind and existence of minimizers is obtained just by imposing the coercivity conditions

$$\frac{1}{C}(|A|^{p_1} + |\operatorname{cof} A|^{p_2} + |\det A|^{p_3}) - C \leq W(x, A), \quad \text{a.e. } x \in \Omega, \quad \text{all } A \in \mathbb{R}^{m \times n},$$

for suitable exponents $p_i \geq 1$, $i \in \{1, 2, 3\}$. This is in agreement with some physical requirements such as the fact that it is needed an infinite amount of energy to reduce something of finite volume to zero volume,

$$W(A) \to \infty \quad \text{when} \quad \det A \to 0,$$

which would be incompatible with the upper bound conditions (required by the quasiconvexity assumption).

The proof of the weak lower semicontinuity of the hyperelastic energy functional assuming polyconvexity goes through the Piola Identity

$$\operatorname{div} \operatorname{cof} Du = 0,$$

a key ingredient in such process.

Well-known references on the application of Calculus of Variations techniques to nonlinear elasticity are again [5, 36, 39, 78, 89].

1.2.3 Γ-convergence

Sometimes we may find ourselves with a family of functionals depending on a parameter, for example, from an approximation argument or fractional or nonlocal framework. As it is common in mathematical analysis, one would like to see if some asymptotic behaviour can be obtained after a limit process applied to functionals. Since we are dealing with variational problems, it would be desirable that the chosen convergence notion satisfies some properties, such as minimizers converging to minimizers. Γ-convergence is designed in such a way that it tries to fulfil that, and other properties, making it the proper notion for the convergence of functionals. For a more detailed explanation see [29, 30, 40].

Definition 1.2.1. *Let X be a metric space and $F_j, F : X \to \mathbb{R}$ be a family of functionals, $j \in \mathbb{N}$. We say that F_j Γ-converges to F as $j \to \infty$ in the strong topology of X if the following two conditions hold:*

- *Liminf inequality: For every family $\{u_j\}_{j\in\mathbb{N}}$ in X such that $u_j \to u$ in X as $j \to \infty$, we have*

$$F(u) \leq \liminf_{j\to\infty} F_j(u_j).$$

- *Limsup inequality: For each $u \in X$, there exists a family $\{u_j\}_{j\in\mathbb{N}} \subset X$ such that $u_j \to u$ in X as $j \to \infty$ and*

$$\limsup_{j\to\infty} F_j(u_j) \leq F(u).$$

Although not in the definition of Γ-convergence a condition zero is required in order to ensure the effectiveness of the Γ-convergence. i.e. a *compactness* property that ensure the existence of converging sequences.

The Γ-convergence can be seen as a generalization of the direct method of the Calculus of Variations, from which has inherit the *lim inf* inequality, which together with the extra compactness requirement, play its role in the existence of minimizers. Then the *lim sup* inequality ensures that the Γ-limit is reached.

A summary of its properties includes the following. It ensures the existence of solution of the problem $\min\{F(u)\,;\, u \in X\}$. Then, we have the convergence of minimum values of F_j to a minimum value of F, accompanied by the already mentioned fact of minimizers converging to minimizers. Other properties include being stable under continuous perturbations and that the Γ limit functional is lower semi-continuous.

1.3 Peridynamics

Peridynamics is a new model of Solid Mechanics proposed by S. Silling in [103] whose goal is to bind together different phenomena in a single framework. Among its motivations we find the fact that classical elasticity models (stated in terms of differential equations and energy functionals), despite having been showed to be quite practical in many situations, stop being valid when singularities appear such as fracture or cavitation (the sudden formation of voids in a material). In order to overcome this, the use of gradients is avoided by computing internal forces by integration instead of differentiation, giving rise to nonlocality as the main difference with classical elasticity [36]. In other words, points separated by a positive distance exert a force upon each other. As a result, the underlying model is a suitable framework where discontinuities may appear naturally, such as fracture, dislocation, or in general, multiscale materials. This would also allow the study of cracks without doing so in a separated way, as is usually done through crack mechanics. In this approach it is also considered that the interaction of points further away than a positive distance δ, called horizon, is negligible. Since the pioneering paper [103], the development of peridynamics has been really overwhelming, both from a theoretical and a numerical-practical point of view. Some references on this are [72, 75, 104, 107] and the two books [62, 77].

Actually, two descriptions of peridynamics were given. The first of them was depicted in [103] and was called bond-based peridynamics. Such approach came with some drawbacks as the fact that it forces the Poisson ratio to be $\nu = \frac{1}{4}$. So as to obtain a more general model, the so-called stated based description was proposed in [105] later on.

1.3.1 Bond-based peridynamics

The original model proposed in [103] was the so-called bond-based model, in which the elastic energy is given by a double integral depending on pairs of points in the reference and deformed configurations. There exists a pair-wise potential function $w : \mathbb{R}^3 \to \mathbb{R}^3$ such that the energy functional is given by

$$E_{BbP} = \int_\Omega \int_{\Omega \cap B(x,\delta)} w(x - x', u(x) - u(x'))\, dx'\, dx,$$

for $\Omega \subset \mathbb{R}^3$ compact, being the reference configuration of a closed, bounded body with reference mass density $\rho : \mathbb{R}^3 \to \mathbb{R}$. This is the term that would represent the energy corresponding to the interaction of the internal forces as opposed to the local version of classical elasticity. Adding the external forces would equal the mass time the acceleration by Newton second law.

We follow the steps in [107] so as to obtain the equations of motion. In this approach, internal forces are modelled through pairs of interactions between points. It is also considered that the interaction of points further away than a positive distance called horizon, δ, is negligible.

Let $y : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^3$ be a motion of Ω, so that $y(x,t)$ is the position at time $t \geq 0$ of a material point $x \in \Omega$. Let also $\Omega_t = y(\Omega, t)$ be the deformed image of Ω at time t. Then, the velocity field is defined as

$$v(x,t) = \dot{y}(x,t) \qquad \text{for every } x \in \Omega, t \geq 0.$$

The main characteristic of bond-based peridynamics is that the term corresponding to the interaction forces is obtained by means of "bonds" between particles. Let $L : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^3$ be the force per unit volume at time t on x due to interactions with other particles in the body. Consequently, if the external body force density field is given by $b : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^3$, the total force vector on a subregion $P \subset \Omega$ is given by

$$\int_P (L + b) dx.$$

Additionally, by Newton third law, this L must be self-equilibrated, i.e.:

$$\int_\Omega L(x,t) dx = 0$$

As nonlocality is the hallmark of the model, instead of using the deformation gradient, as it was aforementioned, (nonlocal) internal interactions are computed through integration of a dual force density $f : \Omega \times \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^3$, whose dimensions are force per unit volume squared. We recall that the dependence on $y(x)$ of the force density functions in [107] is implicit.

$$L(x,t) = \int_\Omega f(x', x, t) dx'; \qquad f(x, x', t) = -f(x', x, t).$$

There also exists a function $\tau : \Omega \times \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^3$, called bond force density, such that $f(x', x, t) = \tau(x', x, t) - \tau(x', x, t)$. Both of them have units of force per unit volume squared.

Hence, now we can write the equations of motion imposing Newton second law

$$\rho(x)\ddot{y}(x,t) = \int_\Omega \tau(x', x, t) - \tau(x, x', t) dx' + b(x,t). \tag{1.6}$$

1.3.2 State-based peridynamics

The initial description of the model suffers from some drawbacks, as previously mentioned, as for example, it forces the Poisson ratio to be $\nu = \frac{1}{4}$. In order to overcome this, S.A. Silling proposed later a reformulation of peridynamics theory called state-based peridynamics. The idea is that with this approach internal forces are not only determined by each pair of bonds, but by all the information of the bonds within each neighbourhood of radius δ. The following is a short introduction of what appears in [107].

We start following the notation used by Silling to describe peridynamic states [107] and then try to adapt it to a notation more consistent with the one used throughout the rest of the document. Firstly, the deformation state is introduced, which apply bonds associated to x, to their deformed images:

$$\begin{aligned} \underline{Y}[x,t] &: H \to \mathbb{R}^3 \quad \text{(vector state)} \\ \underline{Y}[x,t]\langle x'-x\rangle &= y(x',t) - y(x,t) \qquad x, x' \in \Omega, \end{aligned}$$

where H is a neighbourhood of x. Secondly, the force state

$$\begin{aligned} \underline{T}[x,t] &: H \to \mathbb{R}^3 \quad \text{(vector state)} \\ \underline{T}[x,t]\langle x'-x\rangle &= \tau(x',x,t) \qquad x, x' \in \Omega. \end{aligned}$$

Next, we study the constitutive equation, which provides us $\tau(x',x,t)$ according to the deformation. The constitutive equation (for a simple and homogeneous material) determines the force state, whose argument is the deformation state.

$$\underline{T}[x,t] = \underline{\hat{T}}(\underline{Y}[x,t])$$

Written in a more systematic notation, (which is the one we are going to use) what we have is the following

$$\begin{aligned} \tau(x',x,t) &= \underline{T}[x,t]\langle x'-x\rangle = \underline{\hat{T}}(\underline{Y}[x,t])\langle x'-x\rangle \\ &= \hat{\tau}(y(x',t)-y(x,t),x',x) \end{aligned}$$

In this state-based framework, the equations of motion are

$$\rho(x)\ddot{y}(x,t) = \int_\Omega \left(\underline{T}[x,t]\langle x'-x\rangle - \underline{T}[x',t]\langle x-x'\rangle\right) dx' + b(x,t).$$

Elastic peridynamic materials

The analogous model to the hiperelastic one studied in the nonlinear elasticity classic theory is the one considered by Silling as elastic peridynamic material.

Such materials fulfil that there exists a function $W = \hat{W}(\underline{Y})$; $\hat{W} : \mathcal{V} \to \mathbb{R}$ called strain energy density. (As it is indicated in [107], $\mathcal{V}$ denotes the set of all vector states). This energy depends only on $\underline{Y}$, and do not on $\dot{\underline{Y}}$, or θ (absolute temperature). Furthermore, for the ball $B = B(x,r)$, it verifies that

$$\dot{W} = \underline{T} \bullet \dot{\underline{Y}} := \int_B \tau(v-v')dx'.$$

So, we can see that

$$\underline{\hat{T}} = \hat{W}_{\underline{Y}} \qquad \Rightarrow \quad \underline{\hat{T}} = \nabla_y \hat{W}.$$

Therefore, the stationary equation yields

$$0 = \int_B \nabla_y \hat{W}(\underline{Y}[x])\langle x' - x\rangle - \nabla_y \hat{W}(\underline{Y}[x'])\langle x - x'\rangle dx' + b(x). \tag{1.7}$$

Since within the state-based approach each internal force at a point x depends on the collective deformation of all the bonds connected to x in a ball of radius δ, it would be natural to consider the dependence of $\hat{W}$ on $\underline{Y}[x]$ through an integral collecting all the information in a neighbourhood of radius δ of the deformation state $\underline{Y}[x,t]\langle x' - x\rangle = y(x',t) - y(x,t)$ times a kernel with compact support in $B(0,\delta)$ providing information about the mentioned bonds, namely

$$\int_{B(x,\delta)} \frac{y(x) - y(x')}{|x - x'|} \frac{x - x'}{|x - x'|} \rho(x - x')\, dx'. \tag{1.8}$$

Moreover, with this consideration in mind, (1.7) would remind us of the term $\operatorname{div}^s \nabla_y W(u, D^s u)$ or $\operatorname{div}^s_\delta \nabla_y W(u, D^s_\delta u)$ appearing in the Euler-Lagrange equations obtained in Part II and Part III.

Actually, in [81] it was already mentioned that models based on operators like (1.8) would fit in state based peridynamics.

Part I

The nonlinear bond-based peridynamics model

Chapter 2. Bond-based peridynamics fails to recover hyperelasticity

In this chapter, we are concerned with the bond-based model in the general nonlinear situation, and more concretely with its relationship with classical theory of hyperelasticity. A nonlinear model in bond-based peridynamics is determined by a function w, named as *pairwise potential function*, such that the total energy of any deformation $u : \Omega \to \mathbb{R}^m$ of the deformable solid $\Omega \subset \mathbb{R}^n$ is given by

$$\int_\Omega \int_{\Omega \cap B(x,\delta)} w(x - x', u(x) - u(x'))\, dx'\, dx, \tag{2.1}$$

where $\delta > 0$ is the *horizon*, the model parameter which measures the maximum interaction distance between the particles. Physically, $n = m = 3$, but it is of mathematical interest to do the analysis for any n and m, and so it will be done in this chapter. In the isotropic linear elastic case, for which the pairwise potential function is quadratic in its second variable, this model limits the Poisson ratio of homogeneous deformations to be $\frac{1}{4}$, as it is explained in Section 1.3. This shows that the bond-based model suffers from severe restrictions in order to represent a wide variety of elastic materials. The state-based model was proposed as a more general nonlocal peridynamic model that avoids this serious limitation (see subsection 1.3.2). Although the bond-based model presents that restriction, it has been very popular in the last years and has shown to be appropriate and effective in modelling singularity phenomena in situations of practical and academic interest (see the recent survey [72] and the references therein).

In [103], the link between the peridynamic bond-based model and conventional ones was established in terms of the Piola-Kirchhoff stress tensor. In particular, for a given pairwise potential function, there exists a stored-energy density function whose local Piola-Kirchhoff stress tensor coincides with the nonlocal stress tensor (which also measures force per unit area in the reference configuration). In that work, it is also argued that even though there is a stored-energy density verifying such a condition for a given pairwise potential, the reciprocal is not true, i.e., not for every hyperelastic density one can find a pairwise potential function with the same Piola-Kirchhoff tensor. We offer here a different perspective which relies on the energy rather than on the stress. First, we recall that Γ-convergence is the proper concept so as to consider the convergence of variational problems (subsection 1.2.3), and in particular, to tackle the problem of whether the limit of the nonlinear bond-based model (2.1) when $\delta \to 0$ is a local hyperelastic energy like

$$\int_\Omega W(Du(x))\, dx. \tag{2.2}$$

Such question was addressed in [25] in a general setting, showing that the Γ-limit is a vector variational problem, and obtaining an explicit characterization of the Γ-limit (hence, of the W in (2.2)) for a given pairwise potential function w. Now, this chapter is devoted to the results showed in [20], where we pushed forward the calculations in [25], aimed to determine whether it is possible or not to recover typical models of hyperelasticity from bond-based models verifying the natural physical restrictions of frame indifference and isotropy. Our conclusion is that nonlinear bond-based models converge, in the sense of Γ-convergence, to hyperelastic models with very limited structure and degrees of freedom. In particular, they cannot converge to a typical hyperelastic model like Mooney-Rivlin. This result is in agreement with the, previously mentioned, limitations of the bond-based peridynamic model.

Thus, what we show in this chapter is that the nonlinear bond-based peridynamic model suffers from a similar weakness to its linear counterpart. This drawback was hinted at but, to the best of our knowledge, not actually proved. Additionally, we corroborate that the linearization of our limit model requires materials to have a Poisson ratio $\frac{1}{4}$, but through a different path: starting from a nonlinear peridynamic model, we take the limit to arrive at a nonlinear local model and then we linearize it to obtain a linear local model.

Other references dealing with convergence of peridynamics models to local models as the horizon goes to zero are the following. In the nonlinear situation, in [106], the *pointwise* convergence of the state-based peridynamics to classical local models is shown, but the mathematical study in the framework of Γ-convergence is still pending. For the linear case, in [82], the Γ-convergence of linear elastic peridynamics to the local Navier-Lamé system is shown. This work is extended in [83] to the geometrically nonlinear situation. Another reference dealing with the convergence of a nonlocal operator over bounded domains is [26], where it is considered a truncated fractional laplacian in a ball of radius δ, as well as both limits, i.e., when δ goes to zero, recovering the classical operator, and when δ goes to infinity, they recovered the fractional one.

The outline of the chapter is the following. Section 2.1 is devoted to preliminaries, including a summary of the Γ-convergence procedure to pass from nonlocal energies (2.1) to the local energy (2.2) as the horizon δ goes to zero. In Section 2.2, frame indifference and isotropy are imposed to bond-based models, characterizing the pairwise potential densities and giving rise to energies verifying those physical properties. It is also shown that frame indifference and isotropy are preserved when passing to the limit as $\delta \to 0$. In Section 2.3, we perform a preliminar analysis on which stored-energy densities can be recovered when making the Γ-limit as $\delta \to 0$ of bond-based models satisfying frame indifference and isotropy. Finally, in Section 2.4 we show that Mooney-Rivlin models are not recoverable. For this, apart from the analysis of the previous section, we need a property of quasiconvexity theory, which states that a strict quasiconvex function can only be the quasiconvexification of itself.

2.1 Preliminaries

2.1.1 Nonlinear bond-based peridynamics

For a given deformation $u : \Omega \to \mathbb{R}^m$, a general nonlinear energy in the framework of bond-based peridynamics takes the form

$$E_{nl}(u) = \int_\Omega \int_\Omega w(x, x - x', u(x) - u(x'))\, dx'\, dx - \int_\Omega F(x) \cdot u(x)\, dx,$$

[77, 103, 107]. The pairwise potential function $w : \Omega \times \tilde{\Omega} \times \mathbb{R}^m \to \mathbb{R}$, with $\tilde{\Omega} = \Omega - \Omega$ (set of $x - x'$ with $x, x' \in \Omega$), measures the interaction between particles $x, x' \in \Omega$ both in the reference and deformed configurations. As the interaction force between particles is expected to increase as the distance between them decreases, it is natural to assume that the pairwise density $w(x, \cdot, \tilde{y})$ blows up at the origin for each $x \in \Omega$ and $\tilde{y} \in \mathbb{R}^m$. Furthermore, it is also natural to assume that particles separated by a distance bigger than a parameter $\delta > 0$ do not interact at all, so that $w(\cdot, \tilde{x}, \cdot) = 0$ if $|\tilde{x}| > \delta$. The parameter δ is the previously mentioned *horizon* of interaction of particles, and it is a relevant part of the peridynamic model. The application of the direct method of the Calculus of Variations for this type of functionals was studied in [23]. An existence theory was obtained in the Lebesgue L^p spaces under suitable growth conditions on w, whereas the relevant nonlocal convexity notion requires the function

$$y \mapsto \int_{\Omega} w(x, x-x', y-v(x'))\, dx' \tag{2.3}$$

to be convex for a.e. $x \in \Omega$ and any test function $v \in L^p(\Omega, \mathbb{R}^m)$. This condition is actually equivalent, under some technical assumptions, to the weak lower semicontinuity of the functional

$$I_{nl}(u) = \int_{\Omega}\int_{\Omega} w(x, x-x', u(x)-u(x'))\, dx'\, dx \tag{2.4}$$

in $L^p(\Omega, \mathbb{R}^m)$ (see [27,53]). The study of this nonlocal convexity notion, which is strictly weaker than usual convexity of $w(x, \tilde{x}, \cdot)$, has been deepened in [24], including relaxation of functionals lacking this condition (see also [85] for the relaxation).

2.1.2 Passage from nonlocal to local as the horizon goes to zero

It is natural to wonder whether the local energy I in (1.4) can be recovered as the limit of the nonlocal energy I_{nl} in (2.4) when $\delta \to 0$. The right framework to study convergence of variational problems is Γ-convergence (subsection 1.2.3), as, in particular, it implies convergence of minimizers and minimum energies. The Γ-convergence of nonlocal functionals I_{nl} as the horizon tends to zero was studied in [25] in an abstract way. It was shown that under natural assumptions the Γ-limit is a local vector variational problem, and the process to construct such a Γ-limit was explicitly described. The local Γ-limit is recovered in several steps:

i) *Scaling.* Making explicit the dependence of the nonlocal functional with respect to δ, we include a parameter β, that will be clarified below, and scale the functional as

$$I_\delta(u) := \frac{n+\beta}{\delta^{n+\beta}} \int_{\Omega}\int_{\Omega\cap B(x,\delta)} w(x, x-x', u(x)-u(x'))\, dx'\, dx. \tag{2.5}$$

ii) *Blow-up at zero.* We assume $\beta \in \mathbb{R}$ is such that there exists the limit

$$w^{\circ}(x, \tilde{x}, \tilde{y}) := \lim_{t\to 0} \frac{1}{t^{\beta}} w(x, t\tilde{x}, t\tilde{y}), \tag{2.6}$$

for a.e. $x \in \Omega$, and all $\tilde{x} \in \tilde{\Omega}$ and $\tilde{y} \in \mathbb{R}^m$.

iii) *Definition of the local density* $\bar{w}$. We define $\bar{w} : \Omega \times \mathbb{R}^{n\times n} \to \mathbb{R}$ as

$$\bar{w}(x, A) := \int_{\mathbb{S}^{n-1}} w^{\circ}(x, z, Az)\, d\mathcal{H}^{n-1}(z) \qquad x \in \Omega,\ A \in \mathbb{R}^{n\times n},$$

where $\mathbb{S}^{n-1}$ is the n-dimensional unit sphere.

iv) *Quasiconvexification.* The candidate to Γ-limit of I_δ as $\delta \to 0$ is then

$$\tilde{I}(u) := \int_{\Omega} \bar{w}^{qc}(x, Du(x))\, dx,$$

where $w^{qc}(x,\cdot)$ is the quasiconvexification ([39]) of $\bar{w}(x,\cdot)$ defined as

$$\bar{w}^{qc}(x, A) := \sup\{v(x, A) : v(x,\cdot) \le \bar{w}(x,\cdot) \text{ and } v(x,\cdot) \text{ quasiconvex}\}.$$

It is shown in [25] that, under several assumptions on the pairwise potential density w, including the nonlocal convexity (2.3), I_δ Γ-converges to $\tilde{I}$.

We make some comments on the steps i)–ii) of the above procedure. The scaling $\delta^{-(n+\beta)}$ of (2.5) is the only possible one, since it makes that $I_\delta(u) \to \int_\Omega \bar{w}(x, Du)$ for smooth functions u, and any other scaling $f(\delta)$ with $f(\delta) \ll \delta^{-(n+\beta)}$ or $\delta^{-(n+\beta)} \ll f(\delta)$ would make the limit identically zero or identically infinity. The existence of the limit (2.6) and, hence, of β is natural since some models (see, e.g., [54, 93]) based on fractional Sobolev spaces take the form $w(x, \tilde{x}, \tilde{y}) = |\tilde{x}|^{-\alpha}|\tilde{y}|^p$ for some $\alpha, p \in \mathbb{R}$; in this regard, see Example 2.3.2 below.

Our aim in what follows is to check whether typical stored-energy densities in hyperelasticity can be obtained by this procedure. This amounts to asking whether hyperelastic models can be obtained as the Γ-limit of bond-based peridynamics models as the horizon of interaction of particles goes to zero. More concretely, given a polyconvex stored-energy density W, whether there exists a pairwise potential function w such that its corresponding functional I_δ Γ-converges to I as $\delta \to 0$.

2.2 Frame-indifference and isotropy in the bond-based model

In this section we explore how frame indifference and isotropy are translated in mathematical terms into the nonlinear bond-based model. For a pairwise potential function w, frame indifference requires that

$$w(x, \tilde{x}, \tilde{y}) = w(x, \tilde{x}, R\tilde{y}), \qquad \text{a.e. } x \in \Omega, \quad \tilde{x} \in \tilde{\Omega}, \quad \tilde{y} \in \mathbb{R}^m, \quad R \in SO(m). \tag{2.7}$$

We are also interested in isotropic materials, i.e., those whose deformation energy does not depend on the loading, or stretching, direction. Mathematically, this is imposed on the pairwise potential function by requiring

$$w(x,\tilde{x},\tilde{y}) = w(x,R\tilde{x},\tilde{y}), \qquad \text{a.e. } x \in \Omega, \quad \tilde{x} \in \tilde{\Omega}, \quad \tilde{y} \in \mathbb{R}^m, \quad R \in SO(n). \tag{2.8}$$

The next result, which has appeared in the literature before (for instance in [103, 107]), is straightforward.

Proposition 2.2.1. *The bond-based model satisfies frame indifference and isotropy (i.e., the pairwise potential function w satisfies* (2.7) *and* (2.8)*) if and only if there exists $\tilde{w} : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that*

$$w(x,\tilde{x},\tilde{y}) = \tilde{w}(x,|\tilde{x}|,|\tilde{y}|) \qquad \textit{a.e. } x \in \Omega, \quad \tilde{x} \in \tilde{\Omega}, \quad \tilde{y} \in \mathbb{R}^m.$$

An interesting question is whether frame indifference and isotropy are inferred to the Γ-limit obtained as the horizon goes to zero. The answer to this question is positive, as the next result shows.

Proposition 2.2.2. *Given a pairwise potential function w, assume there exists $\beta \in \mathbb{R}$ such that the limit in* (2.6) *exists and the function w° may be defined. If the pairwise potential function w satisfies* (2.7) *and* (2.8)*, then the function W, obtained from w by the procedure described in Section 2.1.2 ($W = \bar{w}^{qc}$), satisfies* (1.1) *(frame indifference)*

$$W(x,RA) = W(x,A) \quad \forall\, x \in \Omega,\ A \in \mathbb{R}^{m\times n}, R \in SO(m)$$

and (1.2) *(isotropy)*

$$W(x,AR) = W(x,A) \quad \forall\, x \in \Omega,\ A \in \mathbb{R}^{m\times n}, R \in SO(m).$$

Proof. We prove frame indifference and isotropy of W all at once, but we emphasize that any of those properties of W is inferred independently from the corresponding property of w. By Proposition 2.2.1, there exists $\tilde{w}$ such that

$$w(x,\tilde{x},\tilde{y}) = \tilde{w}(x,|\tilde{x}|,|\tilde{y}|), \qquad \text{a.e. } x \in \Omega, \quad \tilde{x} \in \tilde{\Omega}, \quad \tilde{y} \in \mathbb{R}^m.$$

By assumption, there exists $\beta \in \mathbb{R}$ such that

$$w^\circ(x,\tilde{x},\tilde{y}) := \lim_{t\to 0} \frac{1}{t^\beta} w(x,t\tilde{x},t\tilde{y}), \qquad \text{a.e. } x \in \Omega, \quad \tilde{x} \in \tilde{\Omega}, \quad \tilde{y} \in \mathbb{R}^m.$$

Then

$$w^\circ(x,\tilde{x},\tilde{y}) = \lim_{t\to 0} \frac{1}{t^\beta} \tilde{w}(x,t|\tilde{x}|,t|\tilde{y}|), \qquad \text{a.e. } x \in \Omega, \quad \tilde{x} \in \tilde{\Omega}, \quad \tilde{y} \in \mathbb{R}^m,$$

and we write, with a small abuse of language, that

$$w^\circ = w^\circ(x, |\tilde{x}|, |\tilde{y}|).$$

Given two rotations $R_1 \in SO(m)$ and $R_2 \in SO(n)$,

$$\begin{aligned}\bar{w}(x, R_1 A R_2) &= \int_{\mathbb{S}^{n-1}} w^\circ(x, |z|, |R_1 A R_2 z|)\, d\mathcal{H}^{n-1}(z) \\ &= \int_{\mathbb{S}^{n-1}} w^\circ(x, |z|, |A R_2 z|)\, d\mathcal{H}^{n-1}(z) \\ &= \int_{\mathbb{S}^{n-1}} w^\circ(x, |R_2^{-1} z|, |Az|)\, d\mathcal{H}^{n-1}(z) \\ &= \int_{\mathbb{S}^{n-1}} w^\circ(x, |z|, |Az|)\, d\mathcal{H}^{n-1}(z) \\ &= \bar{w}(x, A),\end{aligned}$$

hence $\bar{w}$ satisfies (1.1) and (1.2), and, therefore, by [39, Th. 6.14], so does its quasiconvexification $\bar{w}^{qc} = W$. □

2.3 Which local densities can be recovered from nonlocal ones?

In this section we make it explicit the relationship between a pairwise potential function w and the stored-energy function obtained from it by the previous Γ-convergence procedure, in the presence of frame indifference and isotropy. We obtain an explicit identity involving those two functions. In that way, if w is a pairwise potential function such that the corresponding sequence of nonlocal functionals Γ-converges to the local functional given by W, the functions w and W are related by that identity.

Thus, it provides a criterium in order to check whether a local functional with energy density W may be obtained as the Γ-limit of functionals like (2.5). There is no doubt this is interesting from a mathematical point of view, but also from a mechanical perspective, since it permits to answer whether local hyperelastic energies are the Γ-limit of nonlocal bond-based nonlinear models as the horizon goes to zero.

For the next result we consider a pairwise potential function w verifying the frame indifference and isotropy properties, and for simplicity in the exposition we assume that the material is homogeneous, i.e., w does not depend on the material point x, so that $w = w(\tilde{x}, \tilde{y})$. By Proposition 2.2.1, there exists $\tilde{w} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$w(\tilde{x}, \tilde{y}) = \tilde{w}(|\tilde{x}|, |\tilde{y}|), \qquad \tilde{x} \in \tilde{\Omega}, \quad \tilde{y} \in \mathbb{R}^m.$$

Note that step ii) of Section 2.1.2 makes w° a homogeneous function of degree β in the pair $(\tilde{x}, \tilde{y})$, i.e., $w^\circ(x, t\tilde{x}, t\tilde{y}) = t^\beta w^\circ(x, \tilde{x}, \tilde{y})$. Therefore, we may assume, without loss of generality, that w is itself homogeneous of degree β, which amounts to saying that $w = w^\circ$, with w° defined by (2.6). Then, according to Section 2.1.2, the density of the Γ-limit of I_δ is the quasiconvexification of the function

$$\begin{aligned}\bar{w}(A) &= \int_{\mathbb{S}^{n-1}} w(z, Az)\, d\mathcal{H}^{n-1}(z) = \int_{\mathbb{S}^{n-1}} \tilde{w}(|z|, |Az|)\, d\mathcal{H}^{n-1}(z) \\ &= \int_{\mathbb{S}^{n-1}} \tilde{w}(1, |Az|)\, d\mathcal{H}^{n-1}(z).\end{aligned}$$

Notice that the dependence of w on $\tilde{x}$ is irrelevant in order to obtain $\bar{w}$.

The above process motivates the following definition of recoverable function.

Definition 2.3.1. *The function $W : \mathbb{R}^{m\times n} \to \mathbb{R}$ is recoverable if there exist $\bar{w} : \mathbb{R}^{m\times n} \to \mathbb{R}$ and $\tilde{w} : \{1\} \times [0, \infty) \to \mathbb{R}$ such that*

$$\bar{w}(A) = \int_{\mathbb{S}^{n-1}} \tilde{w}(1, |Az|)\, d\mathcal{H}^{n-1}(z), \qquad A \in \mathbb{R}^{m\times n} \tag{2.9}$$

and $W = \bar{w}^{qc}$.

The following result gives a necessary and sufficient condition for a $\bar{w}$ to satisfy (2.9) without invoking $\tilde{w}$. We denote by σ_{n-1} the $\mathcal{H}^{n-1}$ measure of $\mathbb{S}^{n-1}$, and by $⨍$ the mean integral.

Proposition 2.3.1. *The function $\bar{w}$ satisfies (2.9) if and only if for every $A \in \mathbb{R}^{n\times n}$ one has*

$$\bar{w}(A) = ⨍_{\mathbb{S}^{n-1}} \bar{w}(|Az| I)\, d\mathcal{H}^{n-1}(z). \tag{2.10}$$

In this case, a function $\tilde{w}$ giving rise to (2.9) is

$$\tilde{w}(1, t) = \frac{\bar{w}(tI)}{\sigma_{n-1}}, \qquad t \geq 0. \tag{2.11}$$

Proof. Assume $\bar{w}$ satisfies (2.9) for some $\tilde{w}$. Fixed $z \in \mathbb{S}^{n-1}$, by (2.9) we have

$$\begin{aligned}\bar{w}(|Az|I) &= \int_{\mathbb{S}^{n-1}} \tilde{w}(1, \big||Az|Iz'\big|)\, d\mathcal{H}^{n-1}(z') = \int_{\mathbb{S}^{n-1}} \tilde{w}(1, |Az|)\, d\mathcal{H}^{n-1}(z') \\ &= \sigma_{n-1}\tilde{w}(1, |Az|),\end{aligned}$$

hence, combining this with (2.9) we obtain (2.10).

Conversely, assuming that (2.10) holds we define $\tilde{w}$ as (2.11) and we readily obtain (2.9). □

What formula (2.10) says is that $\bar{w}$ is determined just by its values in matrices multiples of the identity.

We implement Proposition 2.3.1 to find several examples of stored-energy functions that come or do not come from a $\tilde{w}$ as in (2.9).

Example 2.3.1. *The functional*

$$I_\delta(u) = \frac{n}{\delta^n} \int_\Omega \int_{\Omega \cap B(x,\delta)} \frac{n}{\sigma_{n-1}} \frac{|u(x) - u(x')|^2}{|x - x'|^2} \, dx' \, dx, \quad u \in L^2(\Omega, \mathbb{R}^m)$$

Γ*-converges as* $\delta \to 0$ *to*

$$I(u) = \int_\Omega |Du(x)|^2 \, dx, \quad u \in H^1(\Omega, \mathbb{R}^m).$$

This assertion is justified by the result of Section 2.1.2 (noting that $\beta = 0$ *in this case), formula* (2.10) *and the following simple computation:*

$$\begin{aligned}\bar{w}(A) &= \int_{\mathbb{S}^{n-1}} \frac{n}{\sigma_{n-1}} |Az|^2 \, d\mathcal{H}^{n-1}(z) = n \sum_{i=1}^m \fint_{\mathbb{S}^{n-1}} \left(\sum_{j=1}^n A_{ij} z_j \right)^2 d\mathcal{H}^{n-1}(z) \\ &= n \sum_{i=1}^m \sum_{j,k=1}^n \fint_{\mathbb{S}^{n-1}} A_{ij} A_{ik} z_j z_k \, d\mathcal{H}^{n-1}(z) = \sum_{i=1}^m \sum_{j=1}^n A_{ij}^2 \\ &= |A|^2.\end{aligned}$$

The following example is a generalization of Example 2.3.1, with different exponents.

Example 2.3.2. *Let* $p > 1$ *and* $\alpha < n + p$*. Then the functional*

$$I_\delta(u) = \frac{n + p - \alpha}{\delta^{n+p-\alpha}} \int_\Omega \int_{\Omega \cap B(x,\delta)} \frac{|u(x) - u(x')|^p}{|x - x'|^\alpha} \, dx' \, dx, \quad u \in L^p(\Omega, \mathbb{R}^m)$$

Γ*-converges as* $\delta \to 0$ *to*

$$I(u) = \int_\Omega \int_{\mathbb{S}^{n-1}} |\nabla u(x) z|^p \, d\mathcal{H}^{n-1}(z) \, dx, \quad u \in W^{1,p}(\Omega, \mathbb{R}^m).$$

Again, this is a consequence of the procedure of Section 2.1.2, since the assumptions of [25] are met. In this case we have $\beta = p - \alpha$ *and the function*

$$\mathbb{R}^{n \times n} \ni A \mapsto \int_{\mathbb{S}^{n-1}} |Az|^p \, d\mathcal{H}^{n-1}(z) \tag{2.12}$$

is convex, hence quasiconvex.

Example 2.3.2 recovers a similar result of [94], in which a limit procedure is done without the device of Γ-convergence. In particular, he shows that the limit energy density (2.12) is independent of α and can be expressed as a function of the principal stretches of A. We also recover that result, since the energy density (2.12) is isotropic.

The following result shows in particular that $|A|^p$ does not satisfy (2.10) for $p \notin \{0, 2\}$, despite the representation (2.12). Indeed, as a consequence of the Examples 2.3.1 and 2.3.3, we have that the only functions of $|A|^2$ that satisfy (2.10) are affine functions of $|A|^2$.

Example 2.3.3. *Let $g : [0, \infty) \to \mathbb{R}$ be a non-affine function of class C^2. Then the function $g(|A|^2)$ does not satisfy (2.10).*

Indeed, assume, for a contradiction, that formula (2.10) holds. Then,

$$g(|A|^2) = \fint_{\mathbb{S}^{n-1}} g\left(\big||Az|I\big|^2\right) d\mathcal{H}^{n-1}(z) = \fint_{\mathbb{S}^{n-1}} g\left(n|Az|^2\right) d\mathcal{H}^{n-1}(z).$$

As g is of class C^2 and non-affine, there exists an interval $I \subset (0, \infty)$ such that $g''|_I > 0$ or $g''|_I < 0$. Suppose that $g''|_I > 0$ and take any A such that the set $\{|Az| \colon z \in \mathbb{S}^{n-1}\}$ contains more than one point, and such that $\{n|Az|^2 \colon z \in \mathbb{S}^{n-1}\} \subset I$. Then, by Jensen's inequality, using that g is strictly convex in $\{n|Az|^2 \colon z \in \mathbb{S}^{n-1}\}$, we obtain

$$\fint_{\mathbb{S}^{n-1}} g\left(n|Az|^2\right) d\mathcal{H}^{n-1}(z) > g\left(n \fint_{\mathbb{S}^{n-1}} |Az|^2 \, d\mathcal{H}^{n-1}(z)\right) = g(|A|^2),$$

which is a contradiction. If $g''|_I < 0$, the inequality above is reversed and we also obtain a contradiction.

The following example shows in particular that $|\operatorname{cof} A|^p$ does not satisfy (2.10) for $p \geq 1$.

Example 2.3.4. *Fix $n = m = 3$. Let $g : [0, \infty) \to \mathbb{R}$ be a convex function such that $\limsup_{t \to \infty} g(t) = \infty$. Then the function $\bar{w}(A) = g(|\operatorname{cof} A|)$ does not satisfy (2.10).*

Indeed, given $A \in \mathbb{R}^{3 \times 3}$ and $z \in \mathbb{S}^2$ we have that $|\mathrm{cof}(|Az|I)| = \sqrt{3}|Az|^2$, so if formula (2.10) holds then, by Jensen's inequality,

$$\begin{aligned} g(|\operatorname{cof} A|) &= \fint_{\mathbb{S}^2} g\left(\sqrt{3}|Az|^2\right) d\mathcal{H}^2(z) \geq g\left(\sqrt{3} \fint_{\mathbb{S}^2} |Az|^2 \, d\mathcal{H}^2(z)\right) \\ &= g\left(\frac{1}{\sqrt{3}}|A|^2\right). \end{aligned}$$

Now fix $\lambda > 0$ and consider A as the matrix with diagonal elements $\lambda, \lambda^{-1}, 1$. Then, the inequality

$$g(|\operatorname{cof} A|) \geq g\left(3^{-1/2}|A|^2\right)$$

reads as

$$g\left(\sqrt{\lambda^2+\lambda^{-2}+1}\right) \geq g\left(3^{-1/2}(\lambda^2+\lambda^{-2}+1)\right),$$

which amounts to saying that

$$g\left(t\right) \geq g\left(3^{-1/2}t^2\right), \qquad t \geq \sqrt{3}.$$

Fix $t_0 \geq \sqrt{3}$. Then

$$\max_{[t_0, 3^{-1/2}t_0^2]} g \geq \max_{[3^{-1/2}t_0^2, 3^{-3/2}t_0^4]} g.$$

Repeating this argument and applying induction we find that

$$\max_{[t_0, 3^{-1/2}t_0^2]} g \geq \max_{\left[t_0, \sqrt{3}\left(\frac{t_0}{\sqrt{3}}\right)^{2^n}\right]} g$$

for all $n \in \mathbb{N}$. Taking $t_0 = 2\sqrt{3}$ we obtain that

$$\max_{[2\sqrt{3}, 4\sqrt{3}]} g \geq \max_{\left[2\sqrt{3}, 2^{2^n}\sqrt{3}\right]} g.$$

Consequently,

$$\max_{[2\sqrt{3}, 4\sqrt{3}]} g \geq \sup_{\left[2\sqrt{3}, \infty\right)} g,$$

which contradicts the assumption $\limsup_{t\to\infty} g(t) = \infty$.

In Example 2.3.4, the convexity hypothesis on g may be relaxed to g being convex on an interval $[a, \infty)$ for some $a > 0$. A similar reasoning can be done with $g(\det A)$, but we postpone to the next section a more definitive result.

We finish this section by showing that the linearization of a recoverable function has a Poisson ratio $\frac{1}{4}$.

Example 2.3.5. *Let $m = n = 3$. Let $\bar{w} : \mathbb{R}^{3\times 3} \to \mathbb{R}$ be the function described in the procedure of Section 2.1.2. Assume that it is of class C^2, comes from a homogeneous isotropic material, $D\bar{w}(I) = 0$ and is quasiconvex. Then its linearization gives rise to a linear elastic material with Poisson's ratio equal to $\frac{1}{4}$. See [36] for a mathematical linear elasticity theory.*

Indeed, by Definition 2.3.1 there exists $g : [0, \infty) \to \mathbb{R}$ *of class* C^2 *such that*

$$\bar{w}(A) = \fint_{\mathbb{S}^2} g(|Az|)\, d\mathcal{H}^2(z), \qquad A \in \mathbb{R}^{3\times 3}. \tag{2.13}$$

Let C *be the elasticity tensor of* $\bar{w}$*. On the one hand, as* $\bar{w}$ *is isotropic,* C *is of the form*

$$Ce \cdot e = 2\mu\, |e|^2 + \lambda\, (\operatorname{tr} e)^2\,, \qquad e \in \mathbb{R}^{3\times 3}_s, \tag{2.14}$$

where $\mathbb{R}^{3\times 3}_s$ *stands for the set of symmetric* $\mathbb{R}^{3\times 3}$ *matrices, and* μ *and* λ *are the Lamé moduli. Poisson's ratio is then calculated through*

$$\nu = \frac{\lambda}{2(\lambda + \mu)}.$$

On the other hand, given $e \in \mathbb{R}^{3\times 3}_s$ *and defining* $f(t) = \bar{w}(I + te)$*, we have that the assumption* $D\bar{w}(I) = 0$ *yields* $f'(0) = 0$*, while* $f''(0) = Ce \cdot e$*. A standard calculation starting from* (2.13) *shows that equality* $f'(0) = 0$ *implies* $g'(1) = 0$*, while*

$$f''(0) = g''(1) \fint_{\mathbb{S}^2} (z \cdot ez)^2\; d\mathcal{H}^2(z). \tag{2.15}$$

From (2.14) *we can see that*

$$\mu = \frac{Ce \cdot e}{2} \qquad \text{for any } e \in \mathbb{R}^{3\times 3}_s \text{ with } \operatorname{tr} e = 0 \text{ and } |e| = 1$$

and

$$3\lambda + 2\mu = \frac{CI \cdot I}{3}.$$

From (2.15) *we compute* $C\bar{e} \cdot \bar{e}$ *for* $\bar{e}$ *being the diagonal matrix with diagonal elements* $\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0$*. We obtain*

$$\begin{aligned} C\bar{e} \cdot \bar{e} &= g''(1) \fint_{\mathbb{S}^2} (z \cdot \bar{e}z)^2\; d\mathcal{H}^2(z) = \frac{g''(1)}{2} \fint_{\mathbb{S}^2} \left(z_1^4 - 2z_1^2 z_2^2 + z_2^4\right) d\mathcal{H}^2(z) \\ &= \frac{2g''(1)}{15}, \end{aligned}$$

where in the last equality we have used the formulas

$$\fint_{\mathbb{S}^2} z_1^4\, d\mathcal{H}^2(z) = \fint_{\mathbb{S}^2} z_2^4\, d\mathcal{H}^2(z) = \frac{1}{5}, \qquad \fint_{\mathbb{S}^2} z_1^2 z_2^2\, d\mathcal{H}^2(z) = \frac{1}{15}$$

(see, e.g., [82, App. A] or [80, App.]). Now, again from (2.15) *we compute*

$$CI \cdot I = g''(1) \fint_{\mathbb{S}^2} |z|^4\; d\mathcal{H}^2(z) = g''(1).$$

With this we obtain that

$$\mu = \frac{g''(1)}{15}, \qquad \lambda = \frac{g''(1)}{15}$$

and, finally,

$$\nu = \frac{1}{4}.$$

The issue of the Poisson ratio $\frac{1}{4}$ in the linear bond-based peridynamic model is well known [103, Sect. 11]. Thus, we have arrived at the same conclusion through a different process. Apart from the fact that we use Γ-convergence, our approach follows the order: we start with a nonlinear peridynamic model, then we take the limit in the horizon to a nonlinear local model, and finally we linearize to obtain a linear local model. In contrast, in references [48, 55, 81, 107] the order is: first a nonlinear peridynamic model, then a linear peridynamic model, and finally a linear local model.

2.4 Mooney-Rivlin materials are not recoverable

By small adaptations of the arguments of Examples 2.3.3 and 2.3.4, one can exhibit large families of stored-energy functions $\bar{w}$ that do not satisfy (2.10). Those examples by themselves do not prove that they are not recoverable. Without the aim of being exhaustive, we present in this section the fact that Mooney-Rivlin materials are not recoverable. In order to do that, we will use the following sufficient condition for which equality $W = \bar{w}$ in the procedure of Section 2.1.2 holds. This result is possibly known for experts in quasiconvexity, but we have not found a reference of it. First we need the definition of strict quasiconvexity. We say that a function $\psi : \mathbb{R}^{m\times n} \to \mathbb{R}$ is *strictly quasiconvex* if and only if

$$\psi(A) < \int_{(0,1)^n} \psi(A + Dv(x))\, dx, \tag{2.16}$$

for all matrices $A \in \mathbb{R}^{m\times n}$ and test functions $v \in C_c^\infty((0,1)^n, \mathbb{R}^m) \setminus \{0\}$.

The definition of strict polyconvexity is as follows. We say that $\psi : \mathbb{R}^{m\times n} \to \mathbb{R}$ is *strictly polyconvex* if there exists a strictly convex function g defined in the set of minors of $\mathbb{R}^{m\times n}$ matrices such that $\psi(A) = g(M(A))$ for all $A \in \mathbb{R}^{m\times n}$, where $M(A)$ is the vector formed by all minors of the matrix A is a given order.

The following sufficient condition for strict quasiconvexity is useful.

Proposition 2.4.1. *Any strictly polyconvex function is strictly quasiconvex.*

Proof. Let $\psi : \mathbb{R}^{m\times n} \to \mathbb{R}$ be strictly polyconvex, and let g be strictly convex such that $\psi(A) = g(M(A))$ for all $A \in \mathbb{R}^{m\times n}$. Fix $A \in \mathbb{R}^{m\times n}$ and $v \in C_c^\infty((0,1)^n, \mathbb{R}^m) \setminus \{0\}$. First of all, since M is quasiaffine then, by a well known result [39],

$$\int_{(0,1)^n} M(A + Dv(x))\, dx = M(A).$$

Now, applying Jensen's inequality, and having into account that g is strictly convex, we have that

$$\int_{(0,1)^n} g(M(A + Dv(x)))\, dx > g\left(\int_{(0,1)^n} M(A + Dv(x))\, dx\right) = g(M(A)).$$

Consequently, ψ is strictly quasiconvex. □

The result that we seek guaranteeing the equality $W = \bar{w}$ is the following.

Proposition 2.4.2. *Let $W : \mathbb{R}^{m\times n} \to \mathbb{R}$ be strictly quasiconvex and let $\bar{w} : \mathbb{R}^{m\times n} \to \mathbb{R}$ be such that $W = \bar{w}^{qc}$. Then $W = \bar{w}$.*

Proof. Our proof is based on the application of gradient Young measures [88]. Let $\mathcal{Y}(A)$ be the set of homogeneous gradient Young measures of barycenter A. It is known that W is strictly quasiconvex if and only if

$$W(A) < \int_{\mathbb{R}^{m\times n}} W(F)\, d\nu(F),$$

for all $A \in \mathbb{R}^{m\times n}$ and all $\nu \in \mathcal{Y}(A) \setminus \{\delta_A\}$, where $\delta_A \in \mathcal{Y}(A)$ is the Dirac delta at A.

By assumption, $W = \bar{w}^{qc} \leq \bar{w}$. Fix $A \in \mathbb{R}^{m\times n}$. By the expression of the quasiconvexification in terms of Young measures ([88]), there exists $\nu \in \mathcal{Y}(A)$, such that

$$\bar{w}^{qc}(A) = \int_{\mathbb{R}^{m\times n}} \bar{w}(F)\, d\nu(F).$$

If $\nu \neq \delta_A$ then

$$\int_{\mathbb{R}^{m\times n}} \bar{w}(F)\, d\nu(F) \geq \int_{\mathbb{R}^{m\times n}} W(F)\, d\nu(F) > W(A),$$

a contradiction with the fact $\bar{w}^{qc} = W$. Therefore, $\nu = \delta_A$ and, hence, $\bar{w}^{qc}(A) = \bar{w}(A)$. □

With all those preliminaries, we are in a position to prove the final result of this section.

Proposition 2.4.3. *Let $n = m = 3$. Let $g : (0,\infty) \to [0,\infty)$ be convex and such that there exists $a > 0$ for which $g|_{[a,\infty)}$ is increasing. Let $\alpha, \beta \geq 0$. Assume that:*

$$\alpha > 0 \quad \text{or} \quad \beta > 0 \quad \text{or} \quad g \text{ is strictly convex.}$$

Suppose, in addition, that

$$\text{if } \beta = 0, \text{ for all } t \geq a \text{ there exists } t_1 > t \text{ with } g(t_1) > g(t). \tag{2.17}$$

Then the function W of (1.3) is not recoverable.

Proof. Assume, for a contradiction, that W is recoverable. We use the notation of Section 2.1.2. The assumptions of α, β and g imply that W is strictly polyconvex. By Proposition 2.4.1, it is strictly quasiconvex, and, in turn, by Proposition 2.4.2 we have that $W = \bar{w}$.

The proof wil be finished as soon as we show that $\bar{w}$ does not satisfy (2.10). For any $A \in \mathbb{R}^{3\times 3}$, using Jensen's inequality we find that

$$\fint_{\mathbb{S}^2} |Az|^4 \, d\mathcal{H}^2(z) \geq \left(\fint_{\mathbb{S}^2} |Az|^2 \, d\mathcal{H}^2(z) \right)^2 = \frac{|A|^4}{9}. \tag{2.18}$$

Now, it is easy to check that the expression

$$\left(\fint_{\mathbb{S}^2} |Az|^3 \, d\mathcal{H}^2(z) \right)^{\frac{1}{3}}$$

defines a norm in $\mathbb{R}^{3\times 3}$. Since all norms are equivalent in $\mathbb{R}^{3\times 3}$, there exists $c > 0$ such that

$$\fint_{\mathbb{S}^2} |Az|^3 \, d\mathcal{H}^2(z) \geq c|A|^3, \qquad A \in \mathbb{R}^{3\times 3}.$$

In fact, we can assume that $c \leq a^{-2}$. Using Jensen's inequality, we find that for all $A \in \mathbb{R}^{3\times 3}$ with $c|A|^3 \geq a$,

$$\fint_{\mathbb{S}^2} g(|Az|^3) \, d\mathcal{H}^2(z) \geq g\left(\fint_{\mathbb{S}^2} |Az|^3 \, d\mathcal{H}^2(z) \right) \geq g(c|A|^3). \tag{2.19}$$

Using (2.18) and (2.19), we have that for all $A \in \mathbb{R}^{3\times 3}$ with $c|A|^3 \geq a$,

$$\fint_{\mathbb{S}^2} \bar{w}(|Az|I) \, d\mathcal{H}^2(z) \geq \alpha|A|^2 + \frac{\beta}{3}|A|^4 + g(c|A|^3).$$

If $\bar{w}$ were recoverable, we would have that, if $c|A|^3 \geq a$,

$$\beta|\operatorname{cof} A|^2 + g(\det A) \geq \frac{\beta}{3}|A|^4 + g(c|A|^3).$$

Let $\lambda > 0$ and let A be the diagonal matrix with diagonal elements $\lambda, 1/\lambda, \sqrt[3]{a/c}$. Then, the inequality above reads as

$$\beta\left(1 + \lambda^2\left(\frac{a}{c}\right)^{\frac{2}{3}} + \frac{1}{\lambda^2}\left(\frac{a}{c}\right)^{\frac{2}{3}}\right) + g\left(\left(\frac{a}{c}\right)^{\frac{1}{3}}\right) \geq$$
$$\frac{\beta}{3}\left(\lambda^2 + \frac{1}{\lambda^2} + \left(\frac{a}{c}\right)^{\frac{2}{3}}\right)^2 + g\left(c\left(\lambda^2 + \frac{1}{\lambda^2} + \left(\frac{a}{c}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}}\right).$$

If $\beta > 0$ then

$$\beta\left(1 + \lambda^2\left(\frac{a}{c}\right)^{\frac{2}{3}} + \frac{1}{\lambda^2}\left(\frac{a}{c}\right)^{\frac{2}{3}}\right) + g\left(\left(\frac{a}{c}\right)^{\frac{1}{3}}\right) \geq \frac{\beta}{3}\left(\lambda^2 + \frac{1}{\lambda^2} + \left(\frac{a}{c}\right)^{\frac{2}{3}}\right)^2,$$

which yields a contradiction when we send $\lambda \to \infty$. If $\beta = 0$ we obtain

$$g\left(\left(\frac{a}{c}\right)^{\frac{1}{3}}\right) \geq g\left(c\left(\lambda^2 + \frac{1}{\lambda^2} + \left(\frac{a}{c}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}}\right),$$

which is again a contradiction due to (2.17), since g is increasing in $[a, \infty)$, and

$$a \leq \left(\frac{a}{c}\right)^{\frac{1}{3}} \leq c\left(\lambda^2 + \frac{1}{\lambda^2} + \left(\frac{a}{c}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

provided that λ is large enough. □

An analogue result holds in the incompressible case.

Proposition 2.4.4. *Let $n = m = 3$. Let $\alpha, \beta \geq 0$. Then the function $W : \mathbb{R}^{3\times 3} \to \mathbb{R} \cup \{\infty\}$ defined by*

$$W(A) = \begin{cases} \alpha|A|^2 + \beta|\operatorname{cof} A|^2, & \textit{if } \det A = 1, \\ \infty & \textit{if } \det A \neq 1 \end{cases}$$

is not recoverable.

Proof. Assume, for a contradiction, that W is recoverable. It is easy to check that the function $g : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ defined by $g(t) = \infty$ for $t \neq 1$ and $g(1) = 0$ is strictly convex at $t = 1$. As in Proposition 2.4.3, W is strictly polyconvex where it is finite, so by Propositions 2.4.1 and 2.4.2, $W = \bar{w}$.

If $\bar{w}$ were not recoverable, by (2.10) we reach a contradiction by considering the matrix A with diagonal elements $\lambda, 1/\lambda, 1$ with $\lambda > 1$, since the right-hand side is infinity, while the left-hand side is finite. □

Using the ideas of Example 2.3.3, one can generalize Proposition 2.4.3 to rule out the recoverability of many families of functions of the style of Mooney–Rivlin, but replacing $|\operatorname{cof} A|^2$ with another convex function of $\operatorname{cof} A$. However, for the sake of simplicity, we have restricted ourselves to a quadratic dependence on $\operatorname{cof} A$.

Part II

Models based on the s-fractional gradient. Fractional energy functionals.

Chapter 3. Bessel Fractional Spaces

The previous chapter showed us that there are some drawbacks concerning the nonlinear bond-based model, in particular, at the time of recovering hyperelastic models. This leads us to consider alternative nonlocal ones. In this sense, in the last years there has been a renewed interest in variational problems involving the so-called *Riesz s-fractional gradient* which, for a function $u \in C_c^\infty(\mathbb{R}^n)$, is defined as

$$D^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} \, dy, \qquad x \in \mathbb{R}^n, \tag{3.1}$$

where $c_{n,s}$ is a suitable constant. Additionally, the s-fractional divergence of a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\operatorname{div}^s \phi(x) = -c_{n,s} \operatorname{pv}_x \int_{\mathbb{R}^n} \frac{\phi(x) + \phi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} \, dy, \tag{3.2}$$

where pv_x stands for the principal value centred at x. These integral operators have attracted an increasing attention in recent years since the publication of [99,100]; (see also references [21,22,37,38,102]). In such references, variational principles for functionals depending on this fractional gradient are addressed, as well as the fractional PDE derived from those as equilibrium equations. The authors consider typical Calculus of Variations problems, with standard growth conditions in which the classical (local) gradient is substituted by $D^s u$. In particular, they studied existence of minimizers of energy functionals like

$$I(u) = \int_{\mathbb{R}^n} W(x, u(x), D^s u(x)) \, dx \tag{3.3}$$

for W a convex function in the scalar case, i.e. when $u : \mathbb{R}^n \to \mathbb{R}$, under the complementary condition of u equals a fixed function outside of a domain Ω. The aim of Part II of this thesis is the study of functional (3.3) but in the notable more difficult vectorial case. More concretely, in Chapter 3 the functional analysis framework for this study is addressed, in Chapter 4 we show an existence theory extending the previous one to the vectorial case with a notion weaker than convexity, called polyconvexity (see Definition 4.0.1). Finally, in Chapter 5 we address the convergence of functional (3.3) to its local counterpart when s goes to 1.

Both $D^s u$ and $\operatorname{div}^s \phi$ are well defined for any $u \in C_c^\infty(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$, respectively, and satisfy the remarkable property of being dual operators in the sense of integration by parts (see Corollary 3.1.2). In [102], the s-fractional gradient together with the s-fractional divergence are studied in a systematic manner. Several important properties are given, such as the uniqueness up to a multiplicative constant of the fractional gradient under natural requirements (invariance under translations and rotations, homogeneity under dilations and some continuity properties in an appropriate functional space), as well as some fractional calculus rules. The results in [102] established, both from a mathematical and physical perspective, what was pointed out earlier in [92, 99, 100], namely, that the s-fractional gradient is the natural definition for a fractional differential object. We agree with the previous references on the claim that the s-fractional gradient deserves more attention in the literature, and likely there will be both more theoretical studies and applications in different contexts. Another reason for which this object deserves attention is the fact that $D^s u$ converges to the classical gradient Du as $s \nearrow 1$. Indeed, for $u \in C_c^\infty(\mathbb{R}^n)$, applying Fourier transform (see Lemma 3.1.7),

$$\widehat{D^s u}(\xi) = \frac{2\pi i \xi}{|2\pi\xi|} |2\pi\xi|^s \, \hat{u}(\xi),$$

which converges to $\widehat{Du}(\xi)$ as $s \nearrow 1$.

This approach based on the fractional gradient would be similar to the *philosophy* of the state-based model of peridynamics (Subsection 1.3.2), but with the significant difference of being defined over the whole space, which makes it less suitable for applications but grants it a rather academic interest. Thus, we could see this as a first "academic" model of nonlocal hyperelasticity, to wit, the fractional case.

This chapter is focused on the study of fractional spaces involving the s-fractional gradient. The previous operator definition allows us to define a functional space with a rather similar structure to classical Sobolev spaces. Actually, it naturally leads to the consideration of the space

$$H^{s,p}(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{H^{s,p}}}$$

under the norm

$$\|u\|_{H^{s,p}(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)} + \|D^s u\|_{L^p(\mathbb{R}^n)}.$$

In fact, this is the notion used in [21, 22, 37, 38]. In this space $H^{s,p}(\mathbb{R}^n)$, the fractional gradient and divergence are defined for general functions in $H^{s,p}(\mathbb{R}^n)$ through a limit process of D^s and div^s, respectively, for compactly supported smooth functions (see Definition 3.1.2 and Lemma 3.1.4).

Interestingly, these spaces were shown to coincide with Bessel potential spaces, introduced in [1,6,7,76]; see also [2, Sect. 7.59–7.66]. The Bessel space $L^{s,p}(\mathbb{R}^n)$ is defined, for $1 < p < \infty$ and $s > 0$, as

$$L^{s,p}(\mathbb{R}^n) := g_s(L^p(\mathbb{R}^n)), \tag{3.4}$$

where the Bessel potential $g_s : \mathbb{R}^n \to \mathbb{R}$ satisfies that its Fourier transform is given by

$$\hat{g}_s(\xi) = (1 + 4\pi^2|\xi|^2)^{-\frac{s}{2}}.$$

In other words, $u \in L^{s,p}(\mathbb{R}^n)$ if and only if u can be written as $u = g_s * f$ for a function $f \in L^p(\mathbb{R}^n)$. This space is usually described by means of the Fourier transform,

$$L^{s,p}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}' \ : \ \mathcal{F}^{-1}\left[(1 + |\xi|^2)^{\frac{s}{2}} \hat{u}\right] \in L^p(\mathbb{R}^n) \right\},$$

equipped with the norm

$$\|u\|_{L^{s,p}} = \|\mathcal{F}^{-1}\left[(1 + |\xi|^2)^{\frac{s}{2}} \hat{u}\right]\|_p. \tag{3.5}$$

Both $\mathcal{F}\varphi$ and $\hat{\varphi}$ stand for the Fourier transform of φ, and $\mathcal{S}'$ is the space of tempered distributions.

As it is clear by the previous definition, Bessel spaces were defined as a generalization of classical Sobolev spaces to a fractional order, albeit the functional spaces which have enjoyed a greater prominence in the fractional literature have been the Gagliardo fractional spaces $W^{s,p}$.

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \ : \ \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} < \infty \right\}$$

for $\Omega \subseteq \mathbb{R}^n$. This can be explained by some of their advantages as they are related to a notion of fractional p-laplacian and their seminorm consists of a double integral which makes it easier to deal with than if there were an absolute value in between (for a broad explanation on these spaces and the fractional laplacian see [46]). Nevertheless, the theory on Bessel spaces is re-emerging since they actually provide structural properties quite useful in the study of variational problems. Furthermore, the relationship between Bessel potential spaces and Riesz fractional gradients comes from [99, Th. 1.7], where it was shown the remarkable fact of identifying $H^{s,p}(\mathbb{R}^n)$ with Bessel spaces, i.e. $H^{s,p}(\mathbb{R}^n) = L^{s,p}(\mathbb{R}^n)$ for any $s \in (0,1)$ and $1 < p < \infty$, with equivalence of the norms $\|\cdot\|_{H^{s,p}}$ and $\|\cdot\|_{L^{s,p}}$. Notice that in [2] the notation $L^{s,p}(\mathbb{R}^n)$ is employed, whereas the same space was denoted by $H^{s,p}(\mathbb{R}^n)$ in [76]; therefore, all in all and on the basis of this equality of spaces, from now on we will refer to $H^{s,p}(\mathbb{R}^n)$ as Bessel fractional spaces whereas we will consider the spaces $W^{s,p}$ as Gagliardo fractional spaces. Moreover, the notation $H^{s,p}(\mathbb{R}^n)$ for Bessel spaces is consistent with its characterization in terms of the Fourier transform, which generalizes the characterization via Fourier transform of Sobolev spaces $W^{k,p}(\mathbb{R}^n)$ to non-integer exponents k (see [67, Sect. 6.2.1] for a detailed discussion on this).

It is also of interest the affine subspace of functions verifying a *complement value condition*; to be precise, given $g \in H^{s,p}(\mathbb{R}^n)$ and a bounded domain $\Omega \subset \mathbb{R}^n$, we consider

$$H^{s,p}_g(\Omega) = \{u \in H^{s,p}(\mathbb{R}^n) \ : \ u = g \text{ in } \Omega^c\},$$

where Ω^c stands for the complement of Ω in $\mathbb{R}^n$. Actually, this is the space used by [99, 100] to search for minimizers of 3.3 under the fundamental hypothesis of convexity of W in the last variable, as well as natural coercivity conditions. Taking advantage of the fractional framework and the properties of Riesz potentials and Fourier transform, they also show some remarkable results on the functional spaces $H^{s,p}$, including a fractional Sobolev-type inequality or the compact embedding into L^p.The book [92, Ch. 15] also pays attention to the s-fractional gradient, providing a proof not based on Fourier transform of a fractional fundamental theorem of calculus, also proved in [99], which is used for showing a Sobolev type inequality from $H^{s,p}$ to L^p. Another reference also dealing with the fractional gradient in the case $p = 1$ is [98], whereas $p = \infty$ is addressed in some of their results in [73]. Actually, in [73] they show the very interesting result of the characterization of the weak lower semicontinuity of functionals like 3.3, and it turned out to be the quasiconvexity of W, i.e. the same property than in the (vectorial) classical case. This assumption of quasiconvexity (weaker than polyconvexity) would also provide the existence of minimizers of functionals like 3.3 in the vectorial case. However, with such assumption some upper bound conditions are also required, which do not fit in hyperelasticity theory where it is assumed that an infinite amount of energy is needed to reduce something of finite volume to zero volume ($W(u, D^s u)$ goes to ∞ when $\det D^s u$ goes to zero). Those upper bound conditions are avoided assuming polyconvexity. It is worth mentioning that case $p = 1$ is intentionally avoided in these chapters. First, because the original statements of the Bessel spaces properties and embeddings (Proposition 3.2.2 and Theorems 3.5.8 and 3.5.14) exclude this case, and, second, because we are concerned with a general existence theory that requires reflexive spaces.

This chapter is devoted to the study of Bessel fractional spaces in a way parallel to any typical introductory exposition of Sobolev spaces (for example [32]), including the functional space structural elements required for the existence theory that follows in Chapters 4 and 5. In the first section of this chapter we first introduce the fractional differential operators D^s and div^s, as well as some properties that will allow us to extend such definitions and results to every function in the Bessel space $H^{s,p}$ where the fractional integration by parts is of major importance. This first section is completed with some interesting formulas where the analogy with the classical case is noticed. This leads to a proper introduction of the functional spaces $H^{s,p}$ in Section 3.2 with some considerations regarding density results also tackled in Section 3.4. For that part some fractional calculus facts are required, which are obtained in Section 3.3 and will also prove to be helpful in Chapters 4 and 5. In Section 3.5 several continuous embeddings from [99] are collected such as Poincaré-Sobolev inequality. It is also completed with some particular results whose proofs is needed in Chapter 5 and an alternative proof of the compact embedding result based on a sort of fractional mean value theorem. Finally, in Section 3.6 some special functions in $H^{s,p}$ not included in $W^{1,p}$ are shown. Those functions exhibit singularities of interest in Solid Mechanics.

Another possibility for the definition of Bessel spaces, also inspired by the classical construction of Sobolev spaces, would be to consider the space defined as the class of $L^p(\mathbb{R}^n)$ functions whose *distributional* s-fractional gradient is also in $L^p(\mathbb{R}^n)$. This approach has been carried out in [33,109]. Given $u \in L^p(\mathbb{R}^n)$, the distributional s-fractional gradient is naturally defined as the distribution $\mathcal{D}^s u$ given by

$$\langle \mathcal{D}^s u, \phi \rangle = -\int u(x) \operatorname{div}^s \phi(x)\, dx, \tag{3.6}$$

for any $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. In fact, in [37] (also collected in [109]) is shown that $C_c^\infty(\mathbb{R}^n)$ is dense in the space

$$\{u \in L^p(\mathbb{R}^n) \ : \ \mathcal{D}^s u \in L^p(\mathbb{R}^n, \mathbb{R}^n)\}$$

equipped with the norm $\|\cdot\|_{H^{s,p}}$, and, consequently,

$$H^{s,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) \ : \ \mathcal{D}^s u \in L^p(\mathbb{R}^n, \mathbb{R}^n)\}$$

for $1 < p < \infty$.

As a third option, the fractional gradient D^s could also lead to the definition of normed spaces by requiring its integrability and that of the function, in the same line as in Sobolev spaces. Moreover, taking into account the definition of Sobolev spaces as the functions whose weak derivative is in L^p, asking for *fractional* integration by parts seems to be a reasonable assumption when working with D^s. Accordingly, we introduce the class

$$\tilde{H}^{s,p}(\mathbb{R}^n) = \left\{ u \in L^p(\mathbb{R}^n) \colon \tilde{D}^s u \in L^p(\mathbb{R}^n, \mathbb{R}^n) \text{ and } u \text{ satisfies (IBP)} \right\},$$

for $\tilde{D}^s$ a extended version of D^s for functions in L^p through principal value, whenever it exists (see Definition 3.1.3). We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ satisfies the integration by parts property (IBP) if $\tilde{D}^s u(x)$ is well defined as a principal value for a.e. $x \in \mathbb{R}^n$ and

$$\int \tilde{D}^s u(x) \cdot \phi(x)\, dx = - \int u(x) \operatorname{div}^s \phi(x)\, dx \tag{IBP}$$

for all $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. We equip $\tilde{H}^{s,p}(\mathbb{R}^n)$ with the norm

$$\|u\|_{\tilde{H}^{s,p}} = \|u\|_p + \|\tilde{D}^s u\|_p.$$

However, although we managed to prove that $C_c^\infty(\mathbb{R}^n)$ is dense in $\tilde{H}^{s,p}(\mathbb{R}^n)$ with an alternative proof to that of [37], it still remains as an open question the fact that if the distributional fractional gradient exists as a function in $L^p(\mathbb{R}^n)$, then so does the fractional gradient given by the principal value definition and both of them coincide. This fact would imply that $\tilde{H}^{s,p}(\mathbb{R}^n)$ is a Banach space and its equivalence with $H^{s,p}(\mathbb{R}^n)$.

In the last decade there has been a great deal of work on fractional PDE of elliptic type involving the fractional Laplacian in some way. Our results here enlarge this theory by giving an existence result of minimizers of nonlinear fractional vector variational problems based on polyconvexity, which implies, in turn, an existence result of solutions to nonlinear fractional PDE systems. The amount of references on nonlocal equations and fractional Laplacian is overwhelming, so for situations related to this work we just cite the survey [96], the paper [100] and the references therein. Concerning fractional spaces, in [38], it is addressed the space of functions u whose total fractional variation is finite, naturally leading to the definition of the space

$BV^s(\mathbb{R}^n)$ and to a s-fractional Caccioppoli perimeter concept. Several interesting results are shown, including a continuous embedding of fractional Sobolev spaces into BV^s, a Sobolev-type inequality, a coarea formula, a s-fractional isoperimetric inequality and a natural s-fractional analogue of De Giorgi's notion of reduced boundary. It is also worth mentioning [57], where both the ill- and well-posedness of the classical Eringen model of nonlocal elasticity are addressed. On the other hand, we would like to point out the relationship of our study with nonlocal elasticity and peridynamics. As mentioned above, the variational principle considered in this PartII is not an appropriate model in solid mechanics, but a version in bounded domains of the functional (3.3) involving a nonlocal gradient similar to (3.1), satisfying additional requirements in order to be physically consistent, fits into the peridynamics state-based model for large deformations [107]. Whereas in $H^{s,p}(\mathbb{R}^n)$, the structural functional analysis facts necessary to prove an existence theory for functionals like (3.3) are known (continuous and compact embeddings into L^p), those were still unknown for an analogous version of this space in bounded domains. In this sense, and since the proof provided for the fractional Piola identity may be adapted in a more or less straightforward way to bounded domains, we think this study may be seen as a first step towards a rigorous mathematical theory of nonlocal hyperelasticity. Furthermore, one primary interest for us is that $H^{s,p}$ is larger than $W^{1,p}$, and functions in $H^{s,p}$ may exhibit singularities prohibited in $W^{1,p}$, as we point out in subsection 3.6. We would like to emphasize that, contrary to classical elasticity, both singularities along points (cavitation) and hypersurfaces are compatible with the existence of solutions in $H^{s,p}$ (see Theorem 4.3.1). This seems to indicate that the L^p norm of $D^s u$ not only contributes to the elastic energy, but also to a kind of surface energy, since the latter is necessary in the modelling of such singularities (see, e.g., [13, 41, 69, 86]).

This Part II encompasses the results from [17, 21, 22].

3.1 Fractional differential operators

We start by stating the definition of the s-fractional gradient and divergence. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$ such that $f \in L^1(B(x, r)^c)$ for every $r > 0$, we define the principal value centered at x of $\int_{\mathbb{R}^n} f$, denoted by

$$\mathrm{pv}_x \int_{\mathbb{R}^n} f \quad \text{or} \quad \mathrm{pv}_x \int f,$$

as

$$\lim_{r \to 0} \int_{B(x,r)^c} f,$$

whenever this limit exists. We have denoted by $B(x,r)$ the open ball centered at x of radius r, and by $B(x,r)^c$ its complement. As most integrals in this chapter (and indeed in the whole Part II of the manuscript) are over $\mathbb{R}^n$, we will use the symbol $\int$ as a substitute for $\int_{\mathbb{R}^n}$.

In order to avoid the principal value in (3.1), we first establish the following definition for C_c^∞ functions and then we extend it by density. The following definitions of s-fractional gradient and divergence are adapted from [21, 84, 99, 100]. Recall that Γ denotes Euler's gamma function.

Definition 3.1.1. *Let* $0 < s < 1$ *and set*

$$c_{n,s} := (n+s-1)\frac{\Gamma\left(\frac{n+s-1}{2}\right)}{\pi^{\frac{n}{2}}\, 2^{1-s}\, \Gamma\left(\frac{1-s}{2}\right)}.$$

a) Let $u \in C_c^\infty(\mathbb{R}^n)$*. We define* $D^s u : \mathbb{R}^n \to \mathbb{R}^n$ *as*

$$D^s u(x) := c_{n,s} \int \frac{u(x)-u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy. \tag{3.7}$$

b) Let $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$*. We define* $\operatorname{div}^s \phi : \mathbb{R}^n \to \mathbb{R}$ *as*

$$\operatorname{div}^s \phi(x) := -c_{n,s} \operatorname{pv}_x \int \frac{\phi(x)+\phi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} dy. \tag{3.8}$$

The integral (3.7) is easily seen to be absolutely convergent for all $x \in \mathbb{R}^n$. Moreover, regarding (3.8), by odd symmetry,

$$-c_{n,s} \operatorname{pv}_x \int \frac{\phi(x)+\phi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} dy = c_{n,s} \int \frac{\phi(x)-\phi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} dy, \tag{3.9}$$

and this last integral is also absolutely convergent. Furthermore, $D^s u \in L^q(\mathbb{R}^n, \mathbb{R}^n)$ and $\operatorname{div}^s \phi \in L^q(\mathbb{R}^n, \mathbb{R}^n)$ for all $q \in [1, \infty]$ for smooth and compactly supported u and ϕ; see Lemma 3.3.1 ([21, Lemma 3.1]), if necessary.

Definition 3.1.1 *a)* naturally extends to vector fields $u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ by replacing (3.7) with

$$D^s u(x) := c_{n,s} \int \frac{u(x)-u(y)}{|x-y|^{n+s}} \otimes \frac{x-y}{|x-y|} dy. \tag{3.10}$$

Here, $\otimes$ stands for the usual tensor product of vectors.

Analogously, if $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is such that its rows satisfy the assumptions of Definition 3.1.1, we denote by $\operatorname{Div}^s M$ the column vector-function whose components are the s-fractional divergences of each row of M.

We now extend Definition 3.1.1 to a broader class of functions.

Definition 3.1.2. *Let* $0 < s < 1$ *and* $1 \leq p < \infty$.

a) Let $u \in L^p(\mathbb{R}^n)$ *be such that there exists a sequence of* $\{u_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ *converging to* u *in* $L^p(\mathbb{R}^n)$ *and for which* $\{D^s u_j\}_{j\in\mathbb{N}}$ *is a Cauchy sequence in* $L^p(\mathbb{R}^n, \mathbb{R}^n)$. *We define* $D^s u$ *as the limit in* $L^p(\mathbb{R}^n, \mathbb{R}^n)$ *of* $D^s u_j$ *as* $j \to \infty$.

b) Let $\phi \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ *be such that there exists a sequence of* $\{\phi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ *converging to* ϕ *in* $L^p(\mathbb{R}^n, \mathbb{R}^n)$ *and for which* $\{\operatorname{div}^s \phi_j\}_{j\in\mathbb{N}}$ *is a Cauchy sequence in* $L^p(\mathbb{R}^n)$. *We define* $\operatorname{div}^s \phi$ *as the limit in* $L^p(\mathbb{R}^n)$ *of* $\operatorname{div}^s \phi_j$ *as* $j \to \infty$.

Of course, for a $u \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, the definition of $D^s u$ is analogous taking into account (3.10). The operators D^s and div^s enjoy a duality property (first shown for smooth functions), which is a fractional integration by parts showed in [21, Theorem 3.6], whose proof follows the lines of [84, Th. 1.4] (a nonlocal integration by parts), (see also [38, 84, 102]). The result from [21, Theorem 3.6] was stated under the following alternative definition for the fractional gradient, which still remains as an open question to see whether it coincides with Definition 3.1.2 and the distributional fractional gradient (3.6), for more general functions, or not. Actually, Theorem 3.1.2 helps to provide a sufficient condition for seeing when such equivalence holds.

Definition 3.1.3. *Let* $u : \mathbb{R}^n \to \mathbb{R}$ *be a measurable function. Let* $0 < s < 1$ *and* $x \in \mathbb{R}^n$, *we define* $\tilde{D}^s u$ *at* x *as*

$$\tilde{D}^s u(x) := c_{n,s} \operatorname{pv}_x \int \frac{u(x) - u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} \, dy,$$

whenever the principal value exists.

Notice that $\tilde{D}^s u = D^s u$ for every $u \in C_c^\infty(\mathbb{R}^n)$. Section 3.4 is devoted to the study of this operator, a functional space based on it, and their relationship with $H^{s,p}(\mathbb{R}^n)$ and Definition 3.1.2.

Theorem 3.1.1. *Let* $0 < s < 1$. *Let* $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *be such that*

$$\int_K \int \frac{|u(x) - u(y)|}{|x-y|^{n+s}} \, dy \, dx < \infty. \tag{3.11}$$

for every compact $K \subset \mathbb{R}^n$. *Then* $\tilde{D}^s u \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ *and for all* $\phi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int \tilde{D}^s u(x) \cdot \phi(x) \, dx = - \int u(x) \operatorname{div}^s \phi(x) \, dx.$$

Proof. Assumption (3.11) implies that $\tilde{D}^s u$ exists a.e. as a Lebesgue integral and $\tilde{D}^s u \in L^1_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$, we have

$$\int \tilde{D}^s u(x) \cdot \phi(x)\, dx = c_{n,s} \int\int \frac{u(x) - u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} \cdot \phi(x)\, dy\, dx. \tag{3.12}$$

On the other hand, as $\phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, by (3.9)

$$-\int u(x) \operatorname{div}^s \phi(x) dx = c_{n,s} \int\int u(x) \frac{\phi(x) + \phi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|}\, dy\, dx. \tag{3.13}$$

Thus, it suffices to establish the equality of the right hand sides of (3.12) and (3.13); in fact, we will establish the equality of the double integrals in the domain $D_\delta := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-y| \geq \delta\}$ for each $\delta > 0$. We have

$$\begin{aligned}&\iint\limits_{D_\delta} \frac{u(x) - u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} \cdot \phi(x)\, dy\, dx = \\ &\iint\limits_{D_\delta} \frac{u(x)\, \phi(x)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|}\, dy\, dx - \iint\limits_{D_\delta} \frac{u(y)\, \phi(x)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|}\, dy\, dx.\end{aligned}$$

If we interchange now the roles of x and y in the second integral, using the symmetry of D_δ, we have

$$-\iint\limits_{D_\delta} \frac{u(y)\, \phi(x)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|}\, dy\, dx = \iint\limits_{D_\delta} \frac{u(x)\, \phi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|}\, dy\, dx,$$

and therefore

$$\iint\limits_{D_\delta} \frac{u(x) - u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} \cdot \phi(x)\, dy\, dx = \iint\limits_{D_\delta} u(x) \frac{\phi(x) + \phi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|}\, dy\, dx,$$

whence the equality of the right hand sides of (3.12) and (3.13) follows. □

In particular, for smooth functions we have the following corollary.

Corollary 3.1.2. *Let $0 < s < 1$. Then, for all $u \in C^\infty_c(\mathbb{R}^n)$ and $\phi \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^n)$ we have*

$$\int D^s u(x) \cdot \phi(x)\, dx = -\int u(x) \operatorname{div}^s \phi(x)\, dx.$$

By taking the constant function $u = 1$ in the previous integration by parts (Theorem 3.1.2) we obtain a sort of fractional divergence theorem. It can also be obtained in a straightforward manner through odd symmetry, as mentioned in [49].

Proposition 3.1.3. *Let $0 < s < 1$, $1 \leq p < \infty$ and $u \in L^p(\mathbb{R}^n)$ with $\operatorname{div}^s u \in L^p(\mathbb{R}^n)$ then*

$$\int_{\mathbb{R}^n} \operatorname{div}^s u(x)\, dx = 0.$$

The following result shows that the above definitions of $D^s u$ and $\operatorname{div}^s \phi$ (Definition 3.1.2) are independent of the sequences $\{u_j\}_{j\in\mathbb{N}}$ and $\{\phi_j\}_{j\in\mathbb{N}}$, respectively, and of the exponent p.

Lemma 3.1.4. *Let $0 < s < 1$ and $1 \leq p, q < \infty$.*

a) Let $u \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ be such that there exist sequences $\{u_j\}_{j\in\mathbb{N}}$ and $\{v_j\}_{j\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^n)$ such that $u_j \to u$ in $L^p(\mathbb{R}^n)$ and $v_j \to u$ in $L^q(\mathbb{R}^n)$, and for which $\{D^s u_j\}_{j\in\mathbb{N}}$ converges to some U in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $\{D^s v_j\}_{j\in\mathbb{N}}$ converges to some V in $L^q(\mathbb{R}^n, \mathbb{R}^n)$. Then $U = V$.

b) Let $\phi \in L^p(\mathbb{R}^n, \mathbb{R}^n) \cap L^q(\mathbb{R}^n, \mathbb{R}^n)$ be such that there exist sequences $\{\phi_j\}_{j\in\mathbb{N}}$ and $\{\theta_j\}_{j\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\phi_j \to u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $\theta_j \to u$ in $L^q(\mathbb{R}^n, \mathbb{R}^n)$, and for which $\{\operatorname{div}^s \phi_j\}_{j\in\mathbb{N}}$ converges to some Φ in $L^p(\mathbb{R}^n)$ and $\{\operatorname{div}^s \theta_j\}_{j\in\mathbb{N}}$ converges to some Θ in $L^q(\mathbb{R}^n, \mathbb{R}^n)$. Then $\Phi = \Theta$.

Proof. We prove *a)*, the proof of *b)* being analogous.

Let $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Then, by Corollary 3.1.2,

$$\int U \cdot \phi = \lim_{j\to\infty} \int D^s u_j \cdot \phi = -\lim_{j\to\infty} \int u_j \operatorname{div}^s \phi = -\int u \operatorname{div}^s \phi$$

and, analogously,

$$\int V \cdot \phi = -\int u \operatorname{div}^s \phi.$$

Thus,

$$\int U \cdot \phi = \int V \cdot \phi$$

for all $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$, whence $U = V$. □

3.1.1 Insightful formulas about the fractional gradient

In this subsection we are going to introduce several relevant results which, from our perspective, could give some insight about the fractional gradient since they might remind us of formulas involving the classical gradient. Most of the fractional literature in the last decade has revolved around a particular operator, the fractional laplacian. The fractional gradient, on the other hand, is certainly less known, however, it turns out that both operators are related through a formula similar to that of the classical case. Next result is from [99, Theorem 1.3].

Theorem 3.1.5. *Let $s \in (0,1)$. If $u \in C_c^\infty(\mathbb{R}^n)$, then*

$$(-\Delta)^s u := \bar{c}_{n,s}\,\mathrm{pv}_x \int \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy = -\sum_{j=1}^n \frac{\partial^s}{\partial x_j^s}\frac{\partial^s}{\partial x_j^s} u = -\operatorname{div}^s D^s u$$

where $\bar{c}_{n,s}$ is a normalizing constant (see [46]) and

$$\frac{\partial^s}{\partial x_j^s} u(x) = D_j^s u(x) := c_{n,s} \int \frac{u(x)-u(y)}{|x-y|^{n+s}}\frac{x_j-y_j}{|x-y|}dy. \tag{3.14}$$

Next, we state a result that can be found in [99, Theorem 1.2] and [92, Lemma 15.9], which says that the fractional gradient can be written as a convolution of the classical one with the Riesz potential. We first recall the definition of Riesz potential. Given $0 < s < n$, the Riesz kernel $I_s : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is

$$I_s(x) = \frac{1}{\gamma(s)}\frac{1}{|x|^{n-s}},$$

where the constant $\gamma(s)$ is given by

$$\gamma(s) = \frac{\pi^{\frac{n}{2}}\,2^s\,\Gamma(\frac{s}{2})}{\Gamma(\frac{n-s}{2})}.$$

The Riesz potential of a locally integrable function f is given by

$$I_s * f(x) = \frac{1}{\gamma(s)}\int \frac{f(y)}{|x-y|^{n-s}}dy.$$

Note the relationship between γ and $c_{n,s}$:

$$c_{n,s} = \frac{n+s-1}{\gamma(1-s)}. \tag{3.15}$$

Formally, the following result can be seen as moving the derivative from one factor of the convolution to the other.

Theorem 3.1.6. *Let $s \in (0,1)$. If $u \in C_c^\infty(\mathbb{R}^n)$, then*

$$D^s u = I_{1-s} * Du.$$

It is interesting to regard the s-fractional gradient from a Fourier analysis perspective. As usual, the Fourier transform of an L^1 function $f : \mathbb{R}^n \to \mathbb{C}$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi}\, dx,$$

and then is extended by duality to the class of tempered distributions. Sometimes we will also use the alternative notation $\mathcal{F}(f) = \hat{f}$. We know that classical differentiation translates, when applying the Fourier transform, into multiplication of the Fourier transform of a function by a monomial. This also happens in a fractional sense in this situation. The following result was proved in [99, Th. 1.4] (but it was mistakenly written with a sign switch); we include here a proof for the reader's convenience, which appears in [22, Lemma 3.1].

Lemma 3.1.7. *Let $0 < s < 1$. Then, for all $u \in C_c^\infty(\mathbb{R}^n)$,*

$$\widehat{D^s u}(\xi) = \frac{2\pi i \xi}{|2\pi\xi|} |2\pi\xi|^s \hat{u}(\xi), \qquad \xi \in \mathbb{R}^n.$$

Proof. By Theorem 3.1.6, $D^s u = I_{1-s} * Du$ for any $u \in C_c^\infty(\mathbb{R}^n)$. We compute the Fourier transform of $D^s u$ in the sense of distributions. We start by checking that $I_{1-s} \in \mathcal{S}'$, where $\mathcal{S}$ is the Schwartz space. Given $\phi \in \mathcal{S}$,

$$\begin{aligned}
&\gamma(1-s)\langle I_{1-s}, \phi\rangle = \int \frac{\phi(x)}{|x|^{n+s-1}} dx \\
&= \int_{B(0,1)} \frac{\phi(x)}{|x|^{n+s-1}} dx + \int_{B(0,1)^c} \phi(x)|x| \frac{1}{|x|^{n+s}} dx \\
&\leq \|\phi\|_\infty \left\| \frac{1}{|x|^{n+s-1}} \right\|_{L^1(B(0,1))} + \|\phi|x|\|_\infty \left\| \frac{1}{|x|^{n+s}} \right\|_{L^1(B(0,1)^c)} \\
&\leq [\|\phi\|_\infty + \|\phi|x|\|_\infty] \left[\left\| \frac{1}{|x|^{n+s-1}} \right\|_{L^1(B(0,1))} + \left\| \frac{1}{|x|^{n+s}} \right\|_{L^1(B(0,1)^c)} \right],
\end{aligned}$$

which shows that the Riesz potential is a continuous linear map over the Schwartz space.

Now, $D^s u \in L^1(\mathbb{R}^n)$, since $u \in C_c^\infty(\mathbb{R}^n)$ (see, e.g., [21, Lemma 3.1] or Lemma 3.3.1) and, so, $D^s u$ can also be regarded as a tempered distribution. Therefore, we can apply the Fourier transform to $D^s u = I_{1-s} * Du$ and, having in mind that the latter is a convolution of the Riesz potential with a Schwartz function, as well as that $\widehat{I_{1-s}}(\xi) = |2\pi\xi|^{-(1-s)}$ (see [110]), we have

$$\widehat{D^s u}(\xi) = \widehat{I_{1-s} * Du}(\xi) = \widehat{I_{1-s}}(\xi)\, \widehat{Du}(\xi) = |2\pi\xi|^{-(1-s)}\widehat{Du}(\xi) = \frac{2\pi i\xi}{|2\pi\xi|^{1-s}}\hat{u}(\xi),$$

for every $\xi \in \mathbb{R}$, as desired. □

We have that not just the Fourier transform of this fractional derivative generalizes that of the classical case but something similar happens with the fractional gradient of a Fourier transform.

Lemma 3.1.8. *Let $0 < s < 1$. Then, for all $u \in C_c^\infty(\mathbb{R}^n)$,*

$$D^s\hat{u} = \mathcal{F}\left(-\frac{2\pi i\xi}{|2\pi\xi|}|2\pi\xi|^s u\right), \qquad \xi \in \mathbb{R}^n.$$

Proof. Arguing as in the proof of Lemma 3.1.7 and having in mind that if f is a radial or even function, then the identity $\mathcal{F}(\hat{f}) = f$ holds, we have that

$$\begin{aligned} D^s\hat{u}(\xi) &= D\hat{u} * I_{1-s}(\xi) = \widehat{-2\pi i\xi u} * \mathcal{F}(\hat{I}_{1-s}) = \mathcal{F}(-2\pi i\xi u\hat{I}_{1-s}) \\ &= \mathcal{F}(-2\pi i\xi u|2\pi\xi|^{1-s}). \end{aligned}$$

□

We end this subsection by showing two uniform bounds on the constant $c_{n,s}$ with respect to s. We denote by ω_n the volume of the unit ball in $\mathbb{R}^n$. We introduce this result here since several other results throughout these chapter would be written with a constant independent of s which will be used in Chapter 5.

Lemma 3.1.9. *Let $n \in \mathbb{N}$. Consider the function $c_{n,\cdot} : [-1,1] \to [0,\infty)$, defined as*

$$c_{n,s} = \begin{cases} \dfrac{\Gamma\left(\frac{n+s+1}{2}\right)}{\pi^{\frac{n}{2}}2^{-s}\Gamma\left(\frac{1-s}{2}\right)} & \text{if } -1 \le s < 1, \\ 0 & \text{if } s = 1. \end{cases}$$

Then

$$\sup_{s\in[-1,1]} c_{n,s} < \infty, \qquad \sup_{s\in[-1,1)} \frac{c_{n,s}}{1-s} < \infty \quad \text{and} \quad \lim_{s\nearrow 1}\frac{c_{n,s}}{1-s} = \frac{1}{\omega_n}.$$

Proof. The function $c_{n,\cdot}$ is clearly continuous in $[-1,1)$. As $\Gamma(z) \to +\infty$ as $z \to 0^+$, we obtain that $c_{n,\cdot}$ is also continuous in $[-1,1]$. Now, using the property $z\Gamma(z) = \Gamma(z+1)$ for $z > 0$, we find that

$$\frac{c_{n,s}}{1-s} = \frac{\Gamma(\frac{n+s+1}{2})}{\pi^{\frac{n}{2}} 2^{1-s} \Gamma(\frac{3-s}{2})},$$

and, hence, the function $s \mapsto \frac{c_{n,s}}{1-s}$ is continuous in $[-1,1)$ and admits a continuous extension to $[-1,1]$. In fact,

$$\lim_{s \nearrow 1} \frac{c_{n,s}}{1-s} = \frac{\Gamma(1+\frac{n}{2})}{\pi^{\frac{n}{2}}} = \frac{1}{\omega_n}.$$

The conclusion follows. □

Note that the definition of $c_{n,\cdot}$ in the previous statement extends that of Definition 3.1.1.

3.2 Fractional spaces

Let $0 < s < 1$, $m \in \mathbb{N}$ and $1 \leq p < \infty$. Given $u \in C_c^\infty(\mathbb{R}^n)$, we define $\|\cdot\|_{H^{s,p}(\mathbb{R}^n,\mathbb{R}^m)}$ as

$$\|u\|_{H^{s,p}(\mathbb{R}^n,\mathbb{R}^m)} = \|u\|_{L^p(\mathbb{R}^n)} + \|D^s u\|_{L^p(\mathbb{R}^{n\times m})},$$

which is easily seen to be a norm. We define the space $H^{s,p}$ as the completion of C_c^∞ under the norm $\|\cdot\|_{H^{s,p}(\mathbb{R}^n,\mathbb{R}^m)}$, and extend accordingly the definition of $\|\cdot\|_{H^{s,p}(\mathbb{R}^n,\mathbb{R}^m)}$ to $H^{s,p}(\mathbb{R}^n,\mathbb{R}^m)$.

$$H^{s,p}(\mathbb{R}^n,\mathbb{R}^m) = \overline{C_c^\infty(\mathbb{R}^n,\mathbb{R}^m)}^{\|\cdot\|_{H^{s,p}}}$$

and we denote $H^{s,p}(\mathbb{R}^n) = H^{s,p}(\mathbb{R}^n,\mathbb{R})$. For the sake of simplicity, we will denote the norm in both $L^p(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n,\mathbb{R}^n)$ by $\|\cdot\|_p$. This is the definition given in [99] (see also [38]).

It is also of interest the affine subspace of functions verifying a *complement value condition*; to be precise, given $g \in H^{s,p}(\mathbb{R}^n)$ and a bounded domain $\Omega \subset \mathbb{R}^n$, we consider

$$H_g^{s,p}(\Omega) = \{u \in H^{s,p}(\mathbb{R}^n) \ : \ u = g \text{ in } \Omega^c\}, \tag{3.16}$$

where Ω^c stands for the complement of Ω in $\mathbb{R}^n$.

The space $H^{s,p}$, together with the s-fractional gradient as a mathematical object, was studied in [99,100] (see also [92, Sect. 15.2]). The first remarkable fact is the identification of $H^{s,p}$ with the classical Bessel potential spaces (see [2,97,110]) established in [99, Th. 1.7].

Theorem 3.2.1. *If* $1 < p < \infty$ *and* $s \in (0,1)$*, then*

$$L^{s,p}(\mathbb{R}^n) = H^{s,p}(\mathbb{R}^n)$$

where $L^{s,p}(\mathbb{R}^n)$ *are defined as in* (3.4)*.*

Thanks to this equivalence, and rewriting well-known properties for Bessel spaces in terms of $H^{s,p}$ spaces, we obtain several basic properties that we summarize in the following proposition (see [2, Ch. 7, p. 221]). We denote by $\hookrightarrow$ continuous inclusion.

Proposition 3.2.2. *Set* $0 < s < 1$ *and* $1 < p < \infty$*. Then:*

a) $C_c^\infty(\mathbb{R}^n)$ *is dense in* $H^{s,p}(\mathbb{R}^n)$*.*

b) $H^{s,p}(\mathbb{R}^n)$ *is reflexive.*

c) If $s < t < 1$ *and* $1 < q \le p \le \frac{nq}{n-(t-s)q}$*, then* $H^{t,q}(\mathbb{R}^n) \hookrightarrow H^{s,p}(\mathbb{R}^n)$*.*

d) If $0 < \mu \le s - \frac{n}{p}$*, then* $H^{s,p}(\mathbb{R}^n) \hookrightarrow C^{0,\mu}(\mathbb{R}^n)$*.*

e) If $p = 2$*, then* $H^{s,2}(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ *with equivalence of norms.*

f) If $0 < s_1 < s < s_2 < 1$ *then* $H^{s_2,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n) \hookrightarrow H^{s_1,p}(\mathbb{R}^n)$*.*

We have denoted by $W^{s,p}$ the classical fractional Sobolev spaces and by $C^{0,\mu}$ the space of Hölder continuous functions of exponent μ.

For a $\phi \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ there is a natural relation between $D^s\phi$ and $\operatorname{div}^s \phi$.

Lemma 3.2.3. *Let* $0 < s < 1$ *and* $1 \le p < \infty$*. Let* $\phi \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$*. Then* $\operatorname{div}^s \phi$ *is well defined and* $\operatorname{tr} D^s\phi = \operatorname{div}^s \phi$ *a.e.*

Proof. Let $\{\phi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be a sequence converging to ϕ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that $\{D^s\phi_j\}_{j\in\mathbb{N}}$ converges to $D^s\phi$ in $L^p(\mathbb{R}^n, \mathbb{R}^{n\times n})$. By linearity, $\operatorname{tr} D^s\phi_j \to \operatorname{tr} D^s\phi$ in $L^p(\mathbb{R}^n)$ as $j \to \infty$. In view of Definition 3.1.2 *b)* and Lemma 3.1.4 *b)*, it suffices to show that $\operatorname{tr} D^s\phi_j = \operatorname{div}^s \phi_j$ for all $j \in \mathbb{N}$. Having in mind that the integrals of (3.10) and of the right hand side of (3.9) are absolutely convergent, we obtain that

$$\begin{aligned}
\operatorname{tr} D^s\phi_j(x) &= c_{n,s} \operatorname{tr}\left(\int \frac{\phi_j(x) - \phi_j(y)}{|x-y|^{n+s}} \otimes \frac{x-y}{|x-y|} dy\right) \\
&= c_{n,s} \int \operatorname{tr}\left(\frac{\phi_j(x) - \phi_j(y)}{|x-y|^{n+s}} \otimes \frac{x-y}{|x-y|}\right) dy \\
&= c_{n,s} \int \frac{\phi_j(x) - \phi_j(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} dy = \operatorname{div}^s \phi_j(x),
\end{aligned}$$

which concludes the proof. □

The integration by parts formula of Corollary 3.1.2 can be extended to $H^{s,p}$ as follows. We denote by p' the conjugate exponent of p.

Proposition 3.2.4. *Let $0 < s < 1$ and $1 < p < \infty$. Then, for all $u \in H^{s,p}(\mathbb{R}^n)$ and $\phi \in H^{s,p'}(\mathbb{R}^n, \mathbb{R}^n)$ we have*

$$\int D^s u(x) \cdot \phi(x)\, dx = -\int u(x) \operatorname{div}^s \phi(x)\, dx.$$

Proof. Let $\{u_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ be a sequence converging to u in $H^{s,p}(\mathbb{R}^n)$, and let $\{\phi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be a sequence converging to ϕ in $H^{s,p'}(\mathbb{R}^n, \mathbb{R}^n)$. Then the following convergences hold as $j \to \infty$:

$$\begin{aligned}
&u_j \to u \text{ in } L^p(\mathbb{R}^n), \quad D^s u_j \to D^s u \text{ in } L^p(\mathbb{R}^n, \mathbb{R}^n),\\
&\phi_j \to \phi \text{ in } L^{p'}(\mathbb{R}^n, \mathbb{R}^n), \quad D^s \phi_j \to D^s \phi \text{ in } L^{p'}(\mathbb{R}^n, \mathbb{R}^{n\times n}),\\
&\operatorname{tr} D^s \phi_j \to \operatorname{tr} D^s \phi \text{ in } L^{p'}(\mathbb{R}^n), \quad \operatorname{div}^s \phi_j \to \operatorname{div}^s \phi \text{ in } L^{p'}(\mathbb{R}^n),
\end{aligned}$$

the last convergence due to Lemma 3.2.3. As a consequence,

$$D^s u_j \cdot \phi_j \to D^s u \cdot \phi \quad \text{and} \quad u_j \operatorname{div}^s \phi_j \to u \operatorname{div}^s \phi \qquad \text{in } L^1(\mathbb{R}^n). \tag{3.17}$$

By Corollary 3.1.2, for each $j \in \mathbb{N}$,

$$\int D^s u_j(x) \cdot \phi_j(x)\, dx = -\int u_j(x) \operatorname{div}^s \phi_j(x)\, dx.$$

This equality and the convergences (3.17) readily imply the conclusion. $\square$

It is natural to consider alternative definitions for spaces based on the fractional gradient. Actually, in [109] it is shown that the space $H^{s,p}(\mathbb{R}^n)$ coincides with the class of $L^p(\mathbb{R}^n)$ functions whose distributional s-fractional gradient (3.6) is also in $L^p(\mathbb{R}^n)$,

$$H^{s,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) \ : \ \mathcal{D}^s u \in L^p(\mathbb{R}^n, \mathbb{R}^n)\}$$

for $1 < p < \infty$ under the same norm $\|\cdot\|_{H^{s,p}}$. In our case, we are also concerned about the identification $H^{s,p}(\mathbb{R}^n) = \tilde{H}^{s,p}(\mathbb{R}^n)$ where

$$\tilde{H}^{s,p}(\mathbb{R}^n) = \left\{ u \in L^p(\mathbb{R}^n) \colon \tilde{D}^s u \in L^p(\mathbb{R}^n, \mathbb{R}^n) \text{ and } u \text{ satisfies (IBP)} \right\},$$

endowed with the norm $\|u\|_{\tilde{H}^{s,p}} = \|u\|_{L^p} + \|\tilde{D}^s u\|_{L^p}$. In $\tilde{H}^{s,p}(\mathbb{R}^n)$ we took the definition for the fractional gradient as a principal value operator given in Definition 3.1.3 where it was mentioned the question of the equality between the fractional gradient operators. The question actually encompasses the equivalence of spaces. We actually show that smooth functions are dense in $\tilde{H}^{s,p}(\mathbb{R}^n)$ (see Section 3.4), although we have left for a future work the fact that if the distributional fractional gradient exists, then, so does the s-fractional gradient version of Definition 3.1.3 and both coincide, in order to have that $\tilde{H}^{s,p}(\mathbb{R}^n)$ is a Banach space and thus, the equivalence with $H^{s,p}(\mathbb{R}^n)$.

3.3 Calculus with the s-fractional gradient $D^s u$

In this section we present some calculus rules which are useful in some computations involving fractional gradients.

We start with a sufficient condition for the s-fractional gradient to be defined everywhere. We denote by $[\varphi]_{C^{0,\alpha}(\mathbb{R}^n)}$ and $[\varphi]_{C^{0,1}(\mathbb{R}^n)}$ the α-Hölder and Lipschitz seminorms of φ, respectively.

Lemma 3.3.1. *Let $0 < \alpha < s < 1$ and $\varphi \in C^{0,\alpha}(\mathbb{R}^n) \cap C^{0,1}(\mathbb{R}^n)$. Then*

$$\sup_{x\in\mathbb{R}^n} \int \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}} dy \leq \frac{\sigma_{n-1}}{1-s} [\varphi]_{C^{0,1}(\mathbb{R}^n)} + \frac{\sigma_{n-1}}{s-\alpha} [\varphi]_{C^{0,\alpha}(\mathbb{R}^n)} < \infty. \tag{3.18}$$

where σ_{n-1} is the area of the unit sphere of $\mathbb{R}^n$.

If, in addition, φ has compact support then $D^s\varphi \in L^r(\mathbb{R}^n)$, for every $r \in [1, \infty]$.

Proof. Let $x \in \mathbb{R}^n$,

$$\begin{aligned} \int \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}} dy &\leq \int_{B(x,1)} \frac{[\varphi]_{C^{0,1}(\mathbb{R}^n)}}{|x-y|^{n+s-1}} dy + \int_{B(x,1)^c} \frac{[\varphi]_{C^{0,\alpha}(\mathbb{R}^n)}}{|x-y|^{n+s-\alpha}} dy \\ &= \int_{B(0,1)} \frac{[\varphi]_{C^{0,1}(\mathbb{R}^n)}}{|z|^{n+s-1}} dz + \int_{B(0,1)^c} \frac{[\varphi]_{C^{0,\alpha}(\mathbb{R}^n)}}{|z|^{n+s-\alpha}} dz \\ &\leq \frac{\sigma_{n-1}}{1-s} [\varphi]_{C^{0,1}(\mathbb{R}^n)} + \frac{\sigma_{n-1}}{s-\alpha} [\varphi]_{C^{0,\alpha}(\mathbb{R}^n)} . \end{aligned}$$

This means that $D^s\varphi \in L^\infty(\mathbb{R}^n)$. Notice that Definition 3.1.1 also holds for Lipschitz functions. Either way this result can also be obtained by density.

Next we are going to see that $D^s\varphi \in L^1(\mathbb{R}^n)$ when φ has compact support. Denote by F the support of φ. Then

$$\int \left| c_{n,s} \int \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy \right| dx \leq |c_{n,s}| \left(A+B\right),$$

where

$$A := \int\int_F \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}} dy\, dx, \qquad B := \int\int_{F^c} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}} dy\, dx.$$

Now, we observe that, applying Fubini's Theorem and (3.18),

$$A = \int_F \int \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}} dx\, dy < \infty. \tag{3.19}$$

We notice that $|\varphi(x)-\varphi(y)| = 0$ for every $(x,y) \in F^c \times F^c$. Therefore, applying again (3.18) we get

$$B = \int_F \int_{F^c} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}} dy\, dx \le \int_F \int \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}} dy\, dx < \infty. \tag{3.20}$$

As a consequence of (3.19) and (3.20), $D^s\varphi \in L^1(\mathbb{R}^n)$. Finally, through a standard interpolation argument, we get that $D^s\varphi \in L^r(\mathbb{R}^n)$ for all $r \in [1,\infty]$. ☐

The proof of Lemma 3.3.1 implies, in particular, that not just $D^s\varphi$ but $\tilde{D}^s\varphi$ is defined everywhere for $\varphi \in C^{0,\alpha}(\mathbb{R}^n) \cap C^{0,1}(\mathbb{R}^n)$ and $0 < \alpha < s < 1$.

The following result defines a nonlocal operator related to the s-fractional gradient.

Lemma 3.3.2. *Let $1 \le q < \infty$ and $0 < \alpha < s < 1$. Let $\varphi \in C^{0,\alpha}(\mathbb{R}^n) \cap C^{0,1}(\mathbb{R}^n)$ and $k \in \mathbb{N}$. Then, the operator $K^s_\varphi : L^q(\mathbb{R}^n, \mathbb{R}^{k\times n}) \to L^q(\mathbb{R}^n, \mathbb{R}^k)$ defined as*

$$K^s_\varphi(U)(x) = c_{n,s} \int \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+s}} U(y) \frac{x-y}{|x-y|} dy, \qquad a.e.\ x \in \mathbb{R}^n,$$

is linear and bounded.

Assume, in addition, that φ has compact support. Then, given $0 < \bar{\alpha} < 1$, there exists a constant $C = C(n,q,\bar{\alpha})$ such that for every $s \in (\bar{\alpha}, 1)$, every $r \in [1,q]$ and $U \in L^q(\mathbb{R}^n, \mathbb{R}^{k\times n})$,

$$\left\| K^s_\varphi(U) \right\|_r \le C \left([\varphi]_{C^{0,\alpha}(\mathbb{R}^n)} + [\varphi]_{C^{0,1}(\mathbb{R}^n)} \right) \|U\|_q . \tag{3.21}$$

Proof. The operator K^s_φ is clearly linear. Let $U \in L^q(\mathbb{R}^n, \mathbb{R}^{k\times n})$. For all $x \in \mathbb{R}^n$ we have

$$\left| K^s_\varphi(U)(x) \right| \le |c_{n,s}| \int \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}} |U(y)|\, dy,$$

so

$$\left|K^s_\varphi(U)(x)\right|^q \leq 2^{q-1}|c_{n,s}|^q\left(g(x)+h(x)\right), \tag{3.22}$$

with

$$g(x) := \left(\int_{B(x,1)} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}}\,|U(y)|\,dy\right)^q \quad \text{and}$$
$$h(x) := \left(\int_{B(x,1)^c} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}}\,|U(y)|\,dy\right)^q.$$

Fix $\alpha\in(0,\bar\alpha)$. Then, applying Hölder's inequality, we get

$$\begin{aligned}
g(x) &\leq [\varphi]^q_{C^{0,1}(\mathbb{R}^n)}\left(\int_{B(x,1)}\frac{|U(y)|}{|x-y|^{n+s-1}}dy\right)^q\\
&= [\varphi]^q_{C^{0,1}(\mathbb{R}^n)}\left(\int_{B(0,1)}\frac{|U(x-z)|}{|z|^{n+s-1}}dz\right)^q\\
&\leq [\varphi]^q_{C^{0,1}(\mathbb{R}^n)}\int_{B(0,1)}\frac{|U(x-z)|^q}{|z|^{n+s-1}}dz\left(\int_{B(0,1)}\frac{1}{|z|^{n+s-1}}dz\right)^{q-1}\\
&= [\varphi]^q_{C^{0,1}(\mathbb{R}^n)}\left(\frac{\sigma_{n-1}}{1-s}\right)^{q-1}\int_{B(0,1)}\frac{|U(x-z)|^q}{|z|^{n+s-1}}dz,
\end{aligned}$$

where σ_{n-1} is the area of the unit sphere of $\mathbb{R}^n$. Integrating,

$$\begin{aligned}
\int g(x)\,dx &\leq [\varphi]^q_{C^{0,1}(\mathbb{R}^n)}\left(\frac{\sigma_{n-1}}{1-s}\right)^{q-1}\int_{B(0,1)}\frac{1}{|z|^{n+s-1}}\int|U(x-z)|^q dx\,dz\\
&= [\varphi]^q_{C^{0,1}(\mathbb{R}^n)}\left(\frac{\sigma_{n-1}}{1-s}\right)^{q}\|U\|_q^q.
\end{aligned} \tag{3.23}$$

As for the term h, applying Hölder's inequality,

$$\begin{aligned}
h(x) &\leq [\varphi]^q_{C^{0,\alpha}(\mathbb{R}^n)}\left(\int_{B(x,1)^c}\frac{|U(y)|}{|x-y|^{n+s-\alpha}}dy\right)^q\\
&\leq [\varphi]^q_{C^{0,\alpha}(\mathbb{R}^n)}\int_{B(0,1)^c}\frac{|U(x-z)|^q}{|z|^{n+s-\alpha}}dz\left(\int_{B(0,1)^c}\frac{1}{|z|^{n+s-\alpha}}dz\right)^{q-1}\\
&= [\varphi]^q_{C^{0,\alpha}(\mathbb{R}^n)}\left(\frac{\sigma_{n-1}}{s-\alpha}\right)^{q-1}\int_{B(0,1)^c}\frac{|U(x-z)|^q}{|z|^{n+s-\alpha}}dz.
\end{aligned}$$

Integrating,

$$\begin{aligned}\int h(x)\,dx &\leq [\varphi]^q_{C^{0,\alpha}(\mathbb{R}^n)}\left(\frac{\sigma_{n-1}}{s-\alpha}\right)^{q-1}\int_{B(0,1)^c}\frac{1}{|z|^{n+s-\alpha}}\int |U(x-z)|^q dx\,dz\\ &= [\varphi]^q_{C^{0,\alpha}(\mathbb{R}^n)}\left(\frac{\sigma_{n-1}}{s-\alpha}\right)^{q}\|U\|^q_q .\end{aligned} \tag{3.24}$$

Putting together (6.9), (6.10) and (3.24) we obtain

$$\left\|K^s_\varphi(U)\right\|^q_q \leq 2^{q-1}|c_{n,s}|^q\left([\varphi]^q_{C^{0,1}(\mathbb{R}^n)}\left(\frac{\sigma_{n-1}}{1-s}\right)^q + [\varphi]^q_{C^{0,\alpha}(\mathbb{R}^n)}\left(\frac{\sigma_{n-1}}{s-\alpha}\right)^q\right)\|U\|^q_q ,$$

so applying Lemma 3.1.9 we find that

$$\left\|K^s_\varphi(U)\right\|_q \leq C\left([\varphi]_{C^{0,\alpha}(\mathbb{R}^n)} + [\varphi]_{C^{0,1}(\mathbb{R}^n)}\right)\|U\|_q \tag{3.25}$$

for some constant C independent of $s \in (\bar{\alpha}, 1)$ and U.

Next, we are going to check the boundedness of $K^s_\varphi : L^q(\mathbb{R}^n, \mathbb{R}^{k\times n}) \to L^1(\mathbb{R}^n, \mathbb{R}^k)$. Denote by F the support of φ. Then

$$\int \left|K^s_\varphi(U)(x)\right| dx \leq |c_{n,s}|\,(A+B)\,, \tag{3.26}$$

where

$$A := \int\int_F \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}}\,|U(y)|\,dy\,dx, \quad B := \int\int_{F^c} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}}\,|U(y)|\,dy\,dx.$$

Now, we observe that, applying Fubini's Theorem, Hölder's inequality and Lemmas 3.3.1 and 3.1.9 there exists $C_0 > 0$ independent of $s \in (\bar{\alpha}, 1)$ such that

$$\begin{aligned}A &\leq \int_F |U(y)|\int \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}}dx\,dy \leq C_0 D_\varphi |F|^{\frac{1}{q'}}\left(\int_F |U(y)|^q\,dy\right)^{\frac{1}{q}}\\ &\leq C_0 D_\varphi |F|^{\frac{1}{q'}}\left\|U\right\|_q ,\end{aligned} \tag{3.27}$$

where, for simplicity, we have denoted

$$D_\varphi := \frac{\sigma_{n-1}}{1-s}\,[\varphi]_{C^{0,1}(\mathbb{R}^n)} + \frac{\sigma_{n-1}}{s-\alpha}\,[\varphi]_{C^{0,\alpha}(\mathbb{R}^n)} .$$

Since $|\varphi(x)-\varphi(y)|=0$ for every $(x,y)\in F^c\times F^c$, in view of Hölder's inequality and Lemma 3.3.1 we get

$$
\begin{aligned}
B &= \int_F\int_{F^c}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}}\,|U(y)|\,dy\,dx\\
&\leq \int_F\left(\int_{F^c}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}}dy\right)^{\frac{1}{q'}}\left(\int_{F^c}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}}\,|U(y)|^q\,dy\right)^{\frac{1}{q}}dx\\
&\leq (C_0D_\varphi)^{\frac{1}{q'}}\int_F\left(\int_{F^c}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}}\,|U(y)|^q\,dy\right)^{\frac{1}{q}}dx.
\end{aligned}
$$

Using again Hölder's inequality, Lemma 3.3.1 and Fubini's Theorem, we obtain

$$
\begin{aligned}
B &\leq (C_0D_\varphi)^{\frac{1}{q'}}\left(\int_F\int_{F^c}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}}\,|U(y)|^q\,dy\,dx\right)^{\frac{1}{q}}|F|^{\frac{1}{q'}}\\
&= (C_0|F|D_\varphi)^{\frac{1}{q'}}\left(\int_{F^c}|U(y)|^q\int_F\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}}dx\,dy\right)^{\frac{1}{q}} \qquad (3.28)\\
&\leq (C_0|F|D_\varphi)^{\frac{1}{q'}}\,(C_0D_\varphi)^{\frac{1}{q}}\left(\int_{F^c}|U(y)|^q\,dy\right)^{\frac{1}{q}}\leq CD_\varphi\,\|U\|_q\,,
\end{aligned}
$$

where $C>0$ is a constant independent of s and U. Inequalities (3.26), (3.27), (3.28) and Lemma 3.1.9 lead us to

$$\left\|K^s_\varphi(U)\right\|_1\leq C\left([\varphi]_{C^{0,\alpha}(\mathbb{R}^n)}+[\varphi]_{C^{0,1}(\mathbb{R}^n)}\right)\|U\|_q\,, \qquad (3.29)$$

for some constant C independent of $s\in(\bar{\alpha},1)$ and U. The conclusion of the theorem is obtained through an interpolation of inequalities (3.25) and (3.29). □

As a consequence of Lemma 3.3.2 and a general result, the operator K^s_φ is continuous from the weak topology of $L^q(\mathbb{R}^n,\mathbb{R}^{k\times n})$ to the weak topology of $L^q(\mathbb{R}^n,\mathbb{R}^k)$ and, in the case of a φ of compact support, from the weak topology of $L^q(\mathbb{R}^n,\mathbb{R}^{k\times n})$ to the weak topology of $L^p(\mathbb{R}^n,\mathbb{R}^k)$ for all $p\in[1,q]$.

The next lemma shows the spaces where the sequence $\{D^su_j\}$ is convergent, provided that $\{u_j\}$ is convergent in $H^{s,p}$. Actually, what this lemma shows is that besides the fact that for $u:\mathbb{R}^n\to\mathbb{R}$ a function with compact support, D^su does not need to have compact support, it can be obtained a fractional equivalent of the fact that if $Du\in L^p(\mathbb{R}^n)$ then $Du\in L^r(\mathbb{R}^n)$ for every $r\in[1,p]$.

Lemma 3.3.3. *Let $0 < s < 1$ and $1 < p < \infty$. Let $u \in C_c^\infty(\mathbb{R}^n)$ and let $\{u_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ be a sequence converging to u in $H^{s,p}(\mathbb{R}^n)$. Assume that there is a compact $K \subset \mathbb{R}^n$ such that $\bigcup_{j=1}^\infty \operatorname{supp} u_j \subset K$. Then $D^s u_j \to D^s u$ in $L^r(\mathbb{R}^n)$ for every $r \in [1, p]$.*

Proof. By linearity, we can assume that $u = 0$. Call $K_B = K + B(0,1)$. Then

$$\|D^s u_j\|_{L^1(\mathbb{R}^n)} \le \|D^s u_j\|_{L^p(K_B)} |K_B|^{\frac{1}{p'}} + \|D^s u_j\|_{L^1(K_B^c)}, \tag{3.30}$$

where $|K_B|$ denotes the Lebesgue measure of K_B, and p' is the conjugate exponent of p.

On the other hand, for every $j \in \mathbb{N}$, we use Fubini's Theorem and Hölder's inequality to get

$$\begin{aligned}
\|D^s u_j\|_{L^1(K_B^c)} &= |c_{n,s}| \int_{K_B^c} \left| \int_K \frac{u_j(x) - u_j(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy \right| dx \\
&\le |c_{n,s}| \int_K \int_{K_B^c} \frac{|u_j(x) - u_j(y)|}{|x-y|^{n+s}}\, dx\, dy \\
&\le |c_{n,s}| \int_K \left(\int_{K_B^c} \frac{|u_j(x) - u_j(y)|^p}{|x-y|^{n+s}}\, dx \right)^{\frac{1}{p}} \left(\int_{K_B^c} \frac{1}{|x-y|^{n+s}}\, dx \right)^{\frac{1}{p'}} dy.
\end{aligned} \tag{3.31}$$

Now, for every $y \in K$ we have $K_B^c - y \subset B(0,1)^c$, so

$$\int_{K_B^c} \frac{1}{|x-y|^{n+s}}\, dx = \int_{K_B^c - y} \frac{1}{|z|^{n+s}}\, dz \le \int_{B(0,1)^c} \frac{1}{|z|^{n+s}}\, dz < \infty.$$

Now, we will use C to denote a constant (depending on n, s and K) which can vary through the proof. So, continuing from (3.31) and applying Hölder's inequality again, we obtain

$$\begin{aligned}
\|D^s u_j\|_{L^1(K_B^c)} &\le C \left(\int_K \int_{K_B^c} \frac{|u_j(x) - u_j(y)|^p}{|x-y|^{n+s}}\, dx dy \right)^{\frac{1}{p}} \le C \|u_j\|_{W^{\frac{s}{p},p}(\mathbb{R}^n)} \\
&\le C \|u_j\|_{H^{s,p}(\mathbb{R}^n)},
\end{aligned}$$

where we have used Proposition 3.2.2 *f)* in the last step. This inequality, together with (3.30), leads to

$$\|D^s u_j\|_{L^1(\mathbb{R}^n)} \le C \|u_j\|_{H^{s,p}(\mathbb{R}^n)} \to 0,$$

by assumption. Finally, through a standard interpolation argument, we obtain the convergence $D^s u_j \to 0$ in $L^r(\mathbb{R}^n)$ for every $r \in [1, p]$. □

Remark 3.3.1. *Later on this manuscript it will be proven that* $H_0^{s,p}(K) = \overline{C_c^\infty(K)}^{\|\cdot\|_{H^{s,p}}}$ *with* K *a compact set but without asking it to be Lipschitz yet (*Step 1 *of Theorem 3.4.8), since* K *coincides with its closure. In that case, Lemma 3.3.3 can be extended by density to functions* $u \in H^{s,p}(\mathbb{R}^n)$ *and* $\{u_j\}_{j\in\mathbb{N}} \subset H^{s,p}(\mathbb{R}^n)$.

Now we introduce a product formula for the s-fractional gradient where we consider first the integral definition 3.1.3. We denote by I the identity matrix of dimension n.

Lemma 3.3.4. *Let* $0 < s < 1$ *and* $1 < p < \infty$. *Let* $g \in L^p(\mathbb{R}^n)$ *be with* $\tilde{D}^s g \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ *and* $\varphi \in C_c^1(\mathbb{R}^n)$. *Then* $\varphi g \in L^p(\mathbb{R}^n)$, $\tilde{D}^s(\varphi g) \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ *and for a.e.* $x \in \mathbb{R}^n$,

$$\tilde{D}^s(\varphi g)(x) = \varphi(x)\tilde{D}^s g(x) + K_\varphi^s(gI)(x).$$

Proof. Clearly $\varphi g \in L^p(\mathbb{R}^n)$. Now, for a.e. $x \in \mathbb{R}^n$ we have

$$\begin{aligned} \tilde{D}^s(\varphi g)(x) &= c_{n,s}\,\mathrm{pv}_x \int \frac{(\varphi g)(x) - (\varphi g)(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy \\ &= c_{n,s}\,\mathrm{pv}_x \int \frac{\varphi(x)g(x) - \varphi(x)g(y) + \varphi(x)g(y) - \varphi(y)g(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy \\ &= \varphi(x)\tilde{D}^s g(x) + K_\varphi^s(gI)(x). \end{aligned}$$

The term $\varphi\, D^s g$ is in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ since $\varphi \in C_c^1(\mathbb{R}^n)$, while the term $K_\varphi^s(gI)$ is in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ by Lemma 3.3.2. □

Notice that depending on the regrouping of the terms, this product formula can be written in different expressions. One of such possibilities is obtained in [38], and in [73] including the case $p = \infty$. As in Lemma 3.3.4, the following result computes the s-fractional divergence of a product.

Lemma 3.3.5. *Let* $0 < s < 1$ *and* $1 < p < \infty$. *Let* $g \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ *be with* $\tilde{D}^s g \in L^p(\mathbb{R}^n, \mathbb{R}^{n\times n})$ *and* $\varphi \in C_c^1(\mathbb{R}^n)$. *Then* $\varphi g \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, $\tilde{D}^s(\varphi g) \in L^p(\mathbb{R}^n, \mathbb{R}^{n\times n})$ *and for a.e.* $x \in \mathbb{R}^n$,

$$-c_{n,s}\,\mathrm{pv}_x \int \frac{\varphi g(x) + \varphi g(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} dy = \varphi(x)\tilde{D}^s g(x) + K_\varphi^s(g^T)(x).$$

Last results are adapted (through density) so as to obtain the Leibniz rule in $H^{s,p}(\mathbb{R}^n)$.

Lemma 3.3.6. *Let $u \in H^{s,p}(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then $\varphi u \in H^{s,p}(\mathbb{R}^n)$,*

$$D^s(\varphi u)(x) = \varphi(x) D^s u(x) + K_\varphi^s(uI)(x),$$

and

$$\operatorname{div}^s(\varphi g)(x) = \varphi(x) \operatorname{div}^s g(x) + K_\varphi^s(g^T)(x).$$

Proof. By definition, $u \in H^{s,p}(\mathbb{R}^n)$ if and only if there exists a sequence $\{u_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $\{u_j\}_{j\in\mathbb{N}}$ converges to u in $L^p(\mathbb{R}^n)$ and $\{D^s u_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R}^n, \mathbb{R}^n)$. It is immediate to check that $\varphi u_j \to \varphi u$ in $L^p(\mathbb{R}^n)$. Let us check that $\{D^s(\varphi u_j)\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R}^n, \mathbb{R}^n)$. Owing to Lemma 3.3.4 we have

$$D^s(\varphi u_j) - D^s(\varphi u_k) = D^s(\varphi(u_j - u_k)) = \varphi D^s(u_j - u_k) + K_\varphi^s(u_j - u_k),$$

with $j, k \in \mathbb{N}$. Since $D^s(u_j - u_k) \to 0$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $j, k \to \infty$, we also have that $\varphi D^s(u_j - u_k) \to 0$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$. By Lemma 3.3.2, since $u_j - u_k \to 0$ in $L^p(\mathbb{R}^n)$ as $j, k \to \infty$, we obtain that $K_\varphi^s(u_j - u_k) \to 0$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$. This shows that $\varphi u \in H^{s,p}(\mathbb{R}^n)$. The Leibniz rule for the s-fractional divergence is obtained analogously from Lemma 3.3.5. □

3.4 Density results

This section is devoted to see if an alternative definition of functional spaces based on the fractional gradient would coincide with $H^{s,p}$. This was already hinted at the introduction and at the end of Section 3.2. In particular, we consider the class

$$\tilde{H}^{s,p}(\mathbb{R}^n) = \left\{ u \in L^p(\mathbb{R}^n) \colon \tilde{D}^s u \in L^p(\mathbb{R}^n, \mathbb{R}^n) \text{ and } u \text{ satisfies (IBP)} \right\},$$

At the moment, the results we have obtained show that every function in $\tilde{H}^{s,p}(\mathbb{R}^n)$ can be obtained as the limit of a sequence of compactly supported smooth functions. However, this just gives us the inclusion $\tilde{H}^{s,p}(\mathbb{R}^n) \subset H^{s,p}(\mathbb{R}^n)$. In order to obtain the reverse one, it is necessary to check that the fractional gradient of every function in $H^{s,p}(\mathbb{R}^n)$ can be written using definition 3.1.3 as well. This coincidence still remains open. The principal difficulty we find is that the fractional gradient $\tilde{D}^s$ is defined in full generality as a principal value.

Theorem 3.4.1. *Set $1 < p < \infty$ and $s \in (0, 1)$. Then, $C_c^\infty(\mathbb{R}^n)$ is dense in $\tilde{H}^{s,p}(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{H^{s,p}}$. Thus,*

$$\overline{C_c^\infty(\mathbb{R}^n) \cap \tilde{H}^{s,p}(\mathbb{R}^n)}^{\|\cdot\|_{H^{s,p}}} = \tilde{H}^{s,p}(\mathbb{R}^n).$$

Remark 3.4.1. *As mentioned in the introduction, in [33, Th. A.1] it was shown the density of compactly supported smooth functions in the class of $L^p(\mathbb{R}^n)$ functions whose s-fractional distributional gradient is in $L^p(\mathbb{R}^n, \mathbb{R}^n)$; as a consequence, [33, Cor. 2.1],*

$$\{u \in L^p(\mathbb{R}^n) \ : \ \mathcal{D}^s u \in L^p(\mathbb{R}^n, \mathbb{R}^n)\} = H^{s,p}(\mathbb{R}^n).$$

Now, whenever $u \in \tilde{H}^{s,p}(\mathbb{R}^n)$, we have $\tilde{D}^s u = \mathcal{D}^s u$ and consequently Theorem 3.4.1 follows as a consequence of [33, Th. A.1]. In any case, the proof provided here differs considerably from that of [33, 109]. We find that it is interesting in its own right, because we work directly with $\tilde{D}^s u$, defined as a principal value. Our approach also leads us to the density result of $H^{s,p}_0(\Omega)$ for a bounded domain Ω.

The proof of Theorem 3.4.1 follows the classical path for proving density of compactly supported smooth functions in $W^{1,p}(\mathbb{R}^n)$, namely, using smoothing by convolution with a sequence of mollifiers and truncation with a sequence of cut-off functions. The main difficulty is that we deal with integral operators involving principal values. We start by defining, for each $0 < s < 1$ and $r > 0$, the operators D^s_r and div^s_r as

$$D^s_r u(x) := c_{n,s} \int_{B(x,r)^c} \frac{u(x) - u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|}\, dy$$

and

$$\operatorname{div}^s_r \phi(x) := -c_{n,s} \int_{B(x,r)^c} \frac{\phi(x) + \phi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|}\, dy.$$

These operators consist of truncated versions of the integrals appearing in the definition of fractional gradient and divergence, respectively. In fact, by definition of principal value, for $u : \mathbb{R}^n \to \mathbb{R}$ measurable

$$\lim_{r \searrow 0} D^s_r u(x) = \tilde{D}^s u(x) \tag{3.32}$$

whenever $\tilde{D}^s u(x)$ exists. Notice that the fractional divergence is going to be applied to smooth functions, so no alternative definition is required. Then, for ϕ a smooth function we obviously have that

$$\lim_{r \searrow 0} \operatorname{div}^s_r \phi(x) = \operatorname{div}^s \phi(x).$$

Lemma 3.4.2. *Let $q \in (1, \infty)$ and $0 < s < 1$. Then there exists a constant $C = C(n, s, q)$ such that*

$$\|\operatorname{div}^s_r \varphi\|_q \leq C \|\varphi\|_{W^{1,q}(\mathbb{R}^n,\mathbb{R}^n)}, \tag{3.33}$$

for any $\varphi \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^n)$ and any $r > 0$.

Proof. We fix $r < 1$ and $x \in \mathbb{R}^n$. Note that, by odd symmetry,

$$\operatorname{div}_r^s \varphi(x) =$$
$$- c_{n,s} \left[\int_{B(x,1)\setminus B(x,r)} \frac{\varphi(y) - \varphi(x)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} dy + \int_{B(x,1)^c} \frac{\varphi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} dy \right],$$

so

$$|\operatorname{div}_r^s \varphi(x)| \le C \left[\int_{B(x,1)} \frac{|\varphi(y) - \varphi(x)|}{|x-y|^{n+s}} \, dy + \int_{B(x,1)^c} \frac{|\varphi(y)|}{|x-y|^{n+s}} \, dy \right], \quad (3.34)$$

where, from now on, $C > 0$ denotes a constant depending on n, s and q whose value may vary from line to line.

Now, by the fundamental theorem of Calculus,

$$|\varphi(y) - \varphi(x)| \le \int_0^1 |D\varphi(x + t(y-x))| \, dt \, |y - x| \,,$$

so, using Fubini's theorem,

$$\begin{aligned} \int_{B(x,1)} \frac{|\varphi(y) - \varphi(x)|}{|x-y|^{n+s}} \, dy &\le \int_{B(x,1)} \frac{\int_0^1 |D\varphi(x + t(y-x))| \, dt}{|x-y|^{n+s-1}} \, dy \\ &= \int_0^1 \int_{B(0,1)} \frac{|D\varphi(x+th)|}{|h|^{n+s-1}} \, dh \, dt. \end{aligned}$$

Then, continuing the inequality in (3.34), we get

$$|\operatorname{div}_r^s \varphi(x)| \le C \left[\int_0^1 \int_{B(0,1)} \frac{|D\varphi(x+th)|}{|h|^{n+s-1}} \, dh \, dt + \int_{B(0,1)^c} \frac{|\varphi(x-h)|}{|h|^{n+s}} \, dh \right].$$

We now apply Hölder's inequality to obtain

$$\begin{aligned} |\operatorname{div}_r^s \varphi(x)| \le C &\left[\int_0^1 \left(\int_{B(0,1)} \frac{1}{|h|^{n+s-1}} \, dh \right)^{\frac{1}{q'}} \left(\int_{B(0,1)} \frac{|D\varphi(x+th)|^q}{|h|^{n+s-1}} \, dh \right)^{\frac{1}{q}} \, dt \right. \\ &\quad \left. + \left(\int_{B(0,1)^c} \frac{1}{|h|^{n+s}} \, dh \right)^{\frac{1}{q'}} \left(\int_{B(0,1)^c} \frac{|\varphi(x-h)|^q}{|h|^{n+s}} \, dh \right)^{\frac{1}{q}} \right] \\ \le C &\left[\int_0^1 \left(\int_{B(0,1)} \frac{|D\varphi(x+th)|^q}{|h|^{n+s-1}} \, dh \right)^{\frac{1}{q}} \, dt \right. \\ &\quad \left. + \left(\int_{B(0,1)^c} \frac{|\varphi(x-h)|^q}{|h|^{n+s}} \, dh \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Now we use the triangular inequality, Jensen's inequality, and Fubini's theorem to obtain

$$\begin{aligned}
\|\operatorname{div}^s_r \varphi\|_q &\leq C \left[\left(\int \int_0^1 \int_{B(0,1)} \frac{|D\varphi(x+th)|^q}{|h|^{n+s-1}} \, dh \, dt \, dx \right)^{\frac{1}{q}} \right. \\
&\qquad \left. + \left(\int \int_{B(0,1)^c} \frac{|\varphi(x-h)|^q}{|h|^{n+s}} \, dh \, dx \right)^{\frac{1}{q}} \right] \\
&= C \left[\left(\int_0^1 \int_{B(0,1)} \frac{\int |D\varphi(x+th)|^q \, dx}{|h|^{n+s-1}} \, dh \, dt \right)^{\frac{1}{q}} \right. \\
&\qquad \left. + \left(\int_{B(0,1)^c} \frac{\int |\varphi(x-h)|^q \, dx}{|h|^{n+s}} \, dh \right)^{\frac{1}{q}} \right] \\
&= C \left[\|D\varphi\|_q \left(\int_{B(0,1)} \frac{1}{|h|^{n+s-1}} \, dh \right)^{\frac{1}{q}} + \|\varphi\|_q \left(\int_{B(0,1)^c} \frac{1}{|h|^{n+s}} \, dh \right)^{\frac{1}{q}} \right],
\end{aligned}$$

which shows (3.33).

When $r \geq 1$, only part of the estimates above are needed. To be precise, we have

$$|\operatorname{div}^s_r \varphi(x)| \leq C \int_{B(x,r)^c} \frac{|\varphi(y)|}{|x-y|^{n+s}} \, dy \leq C \int_{B(x,1)^c} \frac{|\varphi(y)|}{|x-y|^{n+s}} \, dy,$$

and we conclude as before. □

The next lemma establishes that D^s_r and div^s_r are dual operators in the sense of integration by parts. The proof of the lemma is elementary, and is actually contained within the proof of Theorem 3.1.1 (see also [84, Th. 1.4], [21, Th. 3.6]).

Lemma 3.4.3. *Let $p \in [1, \infty]$, $s \in (0,1)$ and $r > 0$. Then, for any $u \in L^p(\mathbb{R}^n)$ we have $D^s_r u \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ and, for each $\varphi \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^n)$,*

$$\int D^s_r u(x) \cdot \varphi(x) \, dx = - \int u(x) \operatorname{div}^s_r \varphi(x) \, dx.$$

Proof. Define

$$C(n,s,r) := \int_{B(0,r)^c} \frac{1}{|h|^{n+s}} \, dh$$

and note that $C(n,s,r) < \infty$. Therefore, given $x \in \mathbb{R}^n$ for which $u(x)$ is finite, we can express

$$D^s_r u(x) = -c_{n,s} \int_{B(x,r)^c} \frac{u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} \, dy, \tag{3.35}$$

so, by Hölder's inequality, if $p < \infty$,

$$\begin{aligned} |D^s_r u(x)| &\le c_{n,s} \int_{B(x,r)^c} \frac{|u(y)|}{|x-y|^{n+s}} \, dy \\ &\le c_{n,s} \, C(n,s,r)^{\frac{1}{p'}} \left(\int_{B(x,r)^c} \frac{|u(y)|^p}{|x-y|^{n+s}} \, dy \right)^{\frac{1}{p}} . \end{aligned}$$

Thus, by Fubini's theorem,

$$\begin{aligned} \|D^s_r u\|_p &\le c_{n,s} \, C(n,s,r)^{\frac{1}{p'}} \left(\int \int_{B(x,r)^c} \frac{|u(y)|^p}{|x-y|^{n+s}} \, dy \, dx \right)^{\frac{1}{p}} \\ &= c_{n,s} \, C(n,s,r) \, \|u\|_p . \end{aligned}$$

If $p = \infty$, we immediately have from (3.35) that $\|D^s_r u\|_p \le c_{n,s} \, C(n,s,r) \, \|u\|_p$.

Therefore, when we define $A_r = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \colon |x-y| \ge r\}$, we have that

$$\int D^s_r u(x) \cdot \varphi(x) \, dx = -c_{n,s} \iint_{A_r} \frac{u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} \, dy \cdot \varphi(x) \, dx \tag{3.36}$$

and this integral is absolutely convergent. Similarly,

$$\operatorname{div}^s_r \varphi(x) = -c_{n,s} \int_{B(x,r)^c} \frac{\varphi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} \, dy$$

and

$$-\int u(x) \operatorname{div}^s_r \varphi(x) \, dx = c_{n,s} \int u(x) \int_{B(x,r)^c} \frac{\varphi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} \, dy \, dx. \tag{3.37}$$

Since we can apply Fubini's theorem, it is then immediate to show that the integrals of (3.36) and (3.37) coincide. $\square$

The next result shows that, if $u \in \tilde{H}^{s,p}(\mathbb{R}^n)$, the a.e. convergences in (3.32) also hold in the sense of distributions.

Lemma 3.4.4. *Let $0 < s < 1$ and $1 < p < \infty$. Then*

$$D_r^s u \rightharpoonup \tilde{D}^s u$$

in the sense of distributions as $r \searrow 0$, for any $u \in \tilde{H}^{s,p}(\mathbb{R}^n)$.

Proof. As a consequence of Lemma 3.4.2 and the a.e. convergence $\operatorname{div}_r^s \varphi \to \operatorname{div}^s \varphi$ as $r \to 0$ (see (3.32)), we have that for any $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$,

$$\operatorname{div}_r^s \varphi \rightharpoonup \operatorname{div}^s \varphi \text{ in } L^{p'}(\mathbb{R}^n).$$

Let $u \in \tilde{H}^{s,p}(\mathbb{R}^n)$. Applying Lemma 3.4.3 and (IBP), we have that

$$\int D_r^s u(x) \cdot \varphi(x)\, dx = -\int u(x) \operatorname{div}_r^s \varphi(x)\, dx$$

converges to

$$-\int u(x) \operatorname{div}^s \varphi(x)\, dx = \int \tilde{D}^s u(x) \cdot \varphi(x)\, dx$$

as $r \searrow 0$, which concludes the proof. □

A crucial fact for proving Theorem 3.4.1, non-trivial in the nonlocal context, is the commutation of the fractional gradient $\tilde{D}^s$ and the convolution. This fact is asserted in the following result for $u \in \tilde{H}^{s,p}(\mathbb{R}^n)$. A similar result concerning a commutation with fractional operators was independently obtained in [37], in the sense of Lipchitz and Bounded variation functions, and measures.

Lemma 3.4.5. *Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be an even function. Then, for any $u \in \tilde{H}^{s,p}(\mathbb{R}^n)$,*

$$\tilde{D}^s(\rho * u) = \rho * \tilde{D}^s u.$$

Proof. We first show that for any $r > 0$,

$$D_r^s(\rho * u) = \rho * D_r^s u. \tag{3.38}$$

Indeed, given $x \in \mathbb{R}^n$ such that $u(x)$ is finite, as in (3.35), we can express

$$D_r^s u(x) = -c_{n,s} \int_{B(x,r)^c} \frac{u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|}\, dy = (d_r^s * u)(x),$$

where $d_r^s : \mathbb{R}^n \to \mathbb{R}^n$ is the function

$$d_r^s(z) = -c_{n,s} \chi_{B(0,r)^c}(z) \frac{1}{|z|^{n+s}} \frac{z}{|z|},$$

which is in $L^1(\mathbb{R}^n, \mathbb{R}^n)$. Since $\rho \in L^1(\mathbb{R}^n)$, we have, thanks to the commutativity and associativity of the convolution, as well as Young's inequality, that

$$D_r^s(\rho * u) = d_r^s * (\rho * u) = \rho * (d_r^s * u) = \rho * D_r^s u,$$

so (3.38) is proved.

Now, by a classical result (e.g., [32, Prop. 4.20]), $\rho * u \in C^\infty(\mathbb{R}^n)$ and $D(\rho * u) = (D\rho) * u$. Consequently, by Young's inequality, $D(\rho * u) \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ and, hence, $\rho * u \in W^{1,p}(\mathbb{R}^n)$. Following the embedding into Sobolev spaces (Proposition 3.2.2 or Proposition 3.5.11 from [22, Prop. 2.7], we have that $\tilde{D}^s(\rho * u) \in L^p(\mathbb{R}^n, \mathbb{R}^n)$; indeed, it is shown there that $\tilde{D}^s v(x)$ is well defined as a Lebesgue integral for a.e. $x \in \mathbb{R}^n$ and $\tilde{D}^s v \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, for any $v \in W^{1,p}(\mathbb{R}^n)$. Furthermore, $\rho * u$ satisfies the conditions of Theorem 3.1.1, and therefore (IBP), hence $\rho * u \in \tilde{H}^{s,p}(\mathbb{R}^n)$. By Lemma 3.4.4,

$$D_r^s(\rho * u) \rightharpoonup \tilde{D}^s(\rho * u) \tag{3.39}$$

in the sense of distributions as $r \to 0$. Again by Lemma 3.4.4, $D_r^s u \rightharpoonup \tilde{D}^s u$ in the sense of distributions. Now let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be an arbitrary test function. Then, by Fubini's theorem and the fact that ρ is even one can easily show (see, e.g., [32, Prop. 4.16]) that

$$\int (\rho * D_r^s u)(x)\varphi(x)\,dx = \int (\rho * \varphi)(x) D_r^s u(x)\,dx$$

and

$$\int (\rho * \tilde{D}^s u)(x)\varphi(x)\,dx = \int (\rho * \varphi)(x)\tilde{D}^s u(x)\,dx.$$

Consequently, as $\rho * \varphi \in C_c^\infty(\mathbb{R}^n)$, we have that

$$\int (\rho * D_r^s u)(x)\varphi(x)\,dx = \int (\rho * \varphi)(x) D_r^s u(x)\,dx$$

converges to

$$\int (\rho * \varphi)(x)\tilde{D}^s u(x) = \int (\rho * \tilde{D}^s u)(x)\varphi(x)\,dx$$

as $r \to 0$. This shows that

$$\rho * D_r^s u \rightharpoonup \rho * \tilde{D}^s u \tag{3.40}$$

in the sense of distributions as $r \to 0$. Comparing (3.38), (3.39) and (3.40) we conclude that $\tilde{D}^s(\rho * u) = \rho * \tilde{D}^s u$ as distributions; since both are L^p functions, the equality also holds a.e. Thus, the lemma is proved. □

Notice that last results is verified, in particular, by functions $u \in C_c^\infty(\mathbb{R}^n)$ for which it is known that $\tilde{D}^s u = Du$. Thus, it can be extended by density so as to obtain the following corollary.

Corollary 3.4.6. *Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be an even function. Then, for any $u \in H^{s,p}(\mathbb{R}^n)$,*

$$D^s(\rho * u) = \rho * D^s u.$$

We are in a position to prove Theorem 3.4.1.

Proof of Theorem 3.4.1. Let us take a standard mollifying sequence $\{\rho_k\}_{k\in\mathbb{N}}$, so that $\rho_k \in C_c^\infty(\mathbb{R}^n)$ is radial, $\operatorname{supp}\rho_k \subset B(0, \frac{1}{k})$, $\rho_k \geq 0$ and $\int \rho_k = 1$. Fix a cut-off function $h \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq h \leq 1$ with $h = 1$ in $B(0,1)$ and $h = 0$ in $B(0,2)^c$. Consider also the cut-off sequence $h_k(x) = h(\frac{x}{k})$ for $k \in \mathbb{N}$.

Let $u \in \tilde{H}^{s,p}(\mathbb{R}^n)$. For each $k \in \mathbb{N}$, define $u_k = \rho_k * u$ and $v_k = h_k\, u_k$. Then $u_k \in C^\infty(\mathbb{R}^n)$ and $v_k \in C_c^\infty(\mathbb{R}^n)$. The proof will be completed as soon as we show that v_k converges to u in the $\|\cdot\|_{H^{s,p}}$ norm. Convergence in L^p is elementary since

$$v_k - u = h_k\,(u_k - u) + (h_k\, u - u)$$

and, hence,

$$\|v_k - u\|_p \leq \|u_k - u\|_p + \|h_k u - u\|_p \to 0 \tag{3.41}$$

as $k \to \infty$ (see, e.g., [32, Th. 4.22]).

For the convergence of the sequence of fractional gradients, by Lemma 3.3.4, we have that

$$\tilde{D}^s v_k = K_{h_k}(u_k) + h_k\, \tilde{D}^s u_k \quad \text{a.e.},$$

where the operator K is as in Lemma 3.3.2. This provides us with the bound

$$\|\tilde{D}^s v_k - \tilde{D}^s u\|_p \leq \|K_{h_k}(u_k)\|_p + \|h_k\, \tilde{D}^s u_k - \tilde{D}^s u\|_p.$$

Recalling from Lemma 3.4.5 that $\tilde{D}^s u_k = \rho_k * \tilde{D}^s u$, the second term in the right hand side of the inequality converges to zero as $k \to \infty$ by the same argument from (3.41). As for the first term, by Lemma 3.3.2,

$$\|K_{h_k}(u_k)\|_p \leq C\left([h_k]_{C^{0,\alpha}(\mathbb{R}^n)} + [h_k]_{C^{0,1}(\mathbb{R}^n)}\right)\|u_k\|_p\,.$$

Now, by Young's inequality, $\|u_k\|_p \leq \|u\|_p$ for all $k \in \mathbb{N}$, while

$$[h_k]_{C^{0,\alpha}(\mathbb{R}^n)} = \frac{1}{k^\alpha}\,[h]_{C^{0,\alpha}(\mathbb{R}^n)} \quad \text{and} \quad [h_k]_{C^{0,1}(\mathbb{R}^n)} = \frac{1}{k}\,[h]_{C^{0,1}(\mathbb{R}^n)}$$

for all $k \in \mathbb{N}$. Therefore, $\|K_{h_k}(u_k)\|_p \to 0$ and, hence, $\|\tilde{D}^s v_k - \tilde{D}^s u\|_p \to 0$ as $k \to \infty$, ending the proof. ∎

In the second part of this section we show the following density result, of interest for applications in fractional variational problems and fractional partial differential equations, where complementary conditions on a given bounded domain are imposed on the admissible functions. We will use the following density result for functions in the Gagliardo fractional spaces $W^{s,p}(\mathbb{R}^n)$, which was given in [31, Prop. B.1].

Proposition 3.4.7. *Let $1 < p < \infty$ and $s \in (0,1)$. Let $\Omega \subset \mathbb{R}^n$ an open, bounded set with a Lipschitz boundary. Then, the space of smooth functions compactly supported in Ω is dense (with respect to the norm $\|\cdot\|_{W^{s,p}}$) in the subspace*

$$W_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^n) \;:\; u = 0 \textit{ in } \Omega^c\}\,.$$

Notice that now we come back to consider the space $H^{s,p}(\mathbb{R}^n)$.

Theorem 3.4.8. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with a Lipschitz boundary, $0 < s < 1$ and $1 < p < \infty$. Then*

$$H_0^{s,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^{s,p}}},$$

where

$$H_0^{s,p}(\Omega) = \{u \in H^{s,p}(\mathbb{R}^n) \;:\; u = 0 \textit{ in } \Omega^c\}\,.$$

Of course, any function in $C_c^\infty(\Omega)$ is also considered as a function in $C_c^\infty(\mathbb{R}^n)$ by extension by zero in Ω^c. The proof of this result relies on two key ingredients. First, the density in $H_0^{s,p}(\Omega)$ of smooth functions supported in $\overline{\Omega}$, which we show following the classical proof for Sobolev functions. Second, the fact that $C_c^\infty(\Omega)$ is dense in $\{u \in W^{s,p}(\mathbb{R}^n) \colon u = 0 \text{ in } \Omega^c\}$ (see Proposition 3.4.7), which has been proved in [31, Prop. B.1].

Proof of Theorem 3.4.8. It is clear that $C_c^\infty(\Omega) \subset H_0^{s,p}(\Omega)$ and that $H_0^{s,p}(\Omega)$ is closed in the $H^{s,p}$ norm. Therefore, it suffices to prove the inclusion $H_0^{s,p}(\Omega) \subset \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^{s,p}}}$. This is done in two steps.

Step 1. We show that given $u \in H^{s,p}(\mathbb{R}^n)$ with $u = 0$ in Ω^c and $\varepsilon > 0$, there exists $\tilde{v} \in C^\infty(\mathbb{R}^n)$ with $\tilde{v} = 0$ in Ω^c such that $\|u - \tilde{v}\|_{H^{s,p}} \leq \frac{\varepsilon}{2}$. The proof of this step follows the line of the analogous result for Sobolev functions, as for instance in [56, Th. 2, Sect. 5.3.2].

We start by defining

$$E_i := \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > \frac{1}{i}\}, \quad i \in \mathbb{N}.$$

Then $\{E_i\}_{i\geq 1}$ is an increasing sequence of open subsets such that $\Omega = \bigcup_{i=1}^{\infty} E_i$. Let us choose $F_0 \subset\subset \Omega$ and set $F_i = E_{i+3} \setminus \bar{E}_i$ $(i \geq 1)$. Then

$$\Omega = \bigcup_{i=0}^{\infty} F_i,$$

so we have expressed Ω as a countable union of open sets compactly contained in Ω. Moreover, that union is locally finite. Now, let $\{\varphi_i\}_{i\in\mathbb{N}}$ be a smooth partition of unity subordinate to the family of open sets $\{F_i\}_{i\in\mathbb{N}}$, that is,

$$\begin{cases} 0 \leq \varphi_i \leq 1,\ \varphi_i \in C_c^{\infty}(F_i), \text{ for all } i \in \mathbb{N}, \\ \sum_{i=0}^{\infty} \varphi = 1 \text{ in } \Omega. \end{cases}$$

For each $i \geq 0$, we have that $\operatorname{supp}(\varphi_i u) \subset F_i$. By Lemma 3.3.6, $\varphi_i u \in H^{s,p}(\mathbb{R}^n)$.

Let η be a standard mollifier: $\eta \in C_c^{\infty}(\mathbb{R}^n)$ is even, $\eta \geq 0$, $\operatorname{supp}\eta \subset B(0,1)$ and $\int \eta = 1$. Then we define the mollifying family

$$\eta_\delta(x) = \frac{1}{\delta^n}\eta\left(\frac{x}{\delta}\right), \qquad x \in \mathbb{R}^n.$$

As a consequence of Corollary 3.4.6 and a standard result for convolutions (see, e.g., [32, Th. 4.22]), there exists $\delta_i > 0$ such that the function $u_i := \eta_{\delta_i} * (\varphi_i u)$ satisfies

$$\|u_i - \varphi_i u\|_p \leq \frac{\varepsilon}{2^{i+3}}, \quad \|D^s u_i - D^s(\varphi_i u)\|_p \leq \frac{\varepsilon}{2^{i+3}}, \quad \operatorname{supp} u_i \subset A_i \tag{3.42}$$

where $A_i = E_{i+4} \setminus \bar{E}_{i-1}$ for $i \geq 1$, and a suitable open set A_0 with $F_0 \subset\subset A_0 \subset\subset \Omega$.

We consider now the function $\tilde{v} : \mathbb{R}^n \to \mathbb{R}$ defined as

$$\tilde{v}(x) := \sum_{i=0}^{\infty} u_i(x).$$

Clearly $\tilde{v} \in C^{\infty}(\mathbb{R}^n)$, as the family $\{A_i\}_{i\geq 0}$ is locally finite. Furthermore, $\tilde{v}(x) = 0$ for all $x \in \Omega^c$.

Since $u = 0$ in Ω^c we can write $u = \sum_{i=0}^{\infty} \varphi_i u$, and using (3.42), we find that

$$\begin{aligned} \|D^s u - D^s \tilde{v}\|_p &= \left\| \sum_{i=0}^{\infty} D^s(\varphi_i u) - \sum_{i=0}^{\infty} D^s u_i \right\|_p \leq \sum_{i=0}^{\infty} \|D^s(\varphi_i u) - D^s u_i\|_p \\ &\leq \sum_{i=0}^{\infty} \frac{\varepsilon}{2^{i+3}} = \frac{\varepsilon}{4}. \end{aligned}$$

The conclusion $\|u-\tilde{v}\|_p \leq \frac{\varepsilon}{4}$ follows from a similar argument. Therefore,

$$\|u-\tilde{v}\|_{H^{s,p}} \leq \frac{\varepsilon}{2}.$$

Step 2. For the function $\tilde{v} \in C^\infty(\mathbb{R}^n)$, with $\tilde{v}=0$ in Ω^c, constructed in the previous step, we claim that there exists $v \in C_c^\infty(\Omega)$ such that $\|v-\tilde{v}\|_{H^{s,p}} \leq \frac{\varepsilon}{2}$, and, consequently,

$$\|u-v\|_{H^{s,p}} \leq \|u-\tilde{v}\|_{H^{s,p}} + \|\tilde{v}-v\|_{H^{s,p}} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and then we would be done.

In order to show the previous claim we first notice that, since $\tilde{v}$ is smooth with bounded support, $\tilde{v} \in W^{t,p}(\mathbb{R}^n)$ for any $t \in (0,1)$. Furthermore, $W^{t,p}(\mathbb{R}^n)$ continuously embeds into $H^{s,p}(\mathbb{R}^n)$ whenever $t > s$ (see Proposition 3.2.2), and therefore there exists a constant $C = C(s,t,p) > 0$ such that

$$\|w\|_{H^{s,p}} \leq C\|w\|_{W^{t,p}}, \quad \text{for all } w \in W^{t,p}(\mathbb{R}^n).$$

Now, having in mind that $\tilde{v}=0$ in Ω^c, by Proposition 3.4.7, there exists $v \in C_c^\infty(\Omega)$ such that

$$\|v-\tilde{v}\|_{W^{t,p}} \leq \frac{\varepsilon}{2C},$$

so

$$\|v-\tilde{v}\|_{H^{s,p}} \leq \frac{\varepsilon}{2}.$$

□

Remark 3.4.2. *The hypothesis of Ω with a Lipschitz boundary comes from [31, Prop. B.1]. However, it remains as an open problem the fact that such hypothesis could be weakened to sets with non-Lipschitz boundaries, in particular, those sets whose boundary is $\bar{\alpha}$-Hölder continuous with $\bar{\alpha} \geq s$.*

3.5 Embeddings of Bessel spaces

Essential tools for obtaining existence of minimizers for variational functionals and other applications are continuous and compact embeddings, such as Sobolev-Poincaré and Morrey type inequalities or Rellich-Kondrachov theorem. The main framework in $H^{s,p}(\mathbb{R}^n)$ was provided by references [99, 100]. We will state such results given therein, in particular, fractional versions of Sobolev, Hardy, Trudinger and Morrey's inequalities together with a fractional compactness result for which we provide an alternative proof based on Frêchet-Kolmogorov theorem (instead of Arcoli-Aszelá theorem as in [100, Theorem 2.2]). To do so we also provide a sort of fractional mean value theorem and some additional proofs as the one for the fractional Poincaré's inequality in order to have inequality constants independent of s.

A crucial fact is the following fractional fundamental theorem of Calculus [38, Th. 3.11] (see also [99, Th. 1.12] or [92, Prop. 15.8]).

Theorem 3.5.1. *Let $0 < s < 1$. For every $u \in C_c^\infty(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$ we have that*

$$u(x) = c_{n,-s} \int D^s u(y) \cdot \frac{x-y}{|x-y|^{n-s+1}} dy.$$

Last result is the fractional counterpart of the following classical representation theorem which can be seen in [63, Lemma 7.14] or [92, Prop. 4.14].

Proposition 3.5.2. *For every $u \in C_c^\infty(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$, we have*

$$u(x) = \frac{1}{\sigma_n} \int_{\mathbb{R}^n} Du(y) \cdot \frac{x-y}{|x-y|^n}\, dy,$$

where σ_n is the area of the unit sphere.

3.5.1 Continuous embeddings

With such tool (Theorem 3.5.1) Shieh and Spector [99, 100] obtained the aforementioned relevant inequalities. In the following, we denote $p_s^* := \frac{np}{n-sp}$. We begin with the statement of a *Fractional Sobolev inequality* proved in [99, Theorem 1.8] for the case $sp < n$.

Theorem 3.5.3. *Let $0 < s < 1$ and $1 < p < \infty$ such that $sp < n$. Then, there exists a constant $C = C(n,p,s) > 0$ such that for every $u \in H^{s,p}(\mathbb{R}^n)$*

$$\|u\|_{L^{p_s^*}(\mathbb{R}^n)} \leq C \|D^s u\|_{L^p(\mathbb{R}^n)}.$$

Corollary 3.5.4. *Let $0 < s < 1$, $1 < p < \infty$ and $u \in H^{s,p}(\mathbb{R}^n)$. If $D^s u = 0$ on $\mathbb{R}^n$, then $u = 0$ on $\mathbb{R}^n$.*

For the same case, $sp < n$, they also provide a *Fractional Hardy inequality* (see [99, Theorem 1.9]).

Theorem 3.5.5. *Let $0 < s < 1$ and $1 < p < \infty$ such that $sp < n$. Then, there exists a constant $C = C(n,p,s) > 0$ such that for every $u \in H^{s,p}(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}}\, dx \leq C \int_{\mathbb{R}^n} |D^s u(x)|^p\, dx.$$

Following with the critical case, [99, Theorem 1.10] also provided a *Fractional Trudinger inequality.*

Theorem 3.5.6. *Let $0 < s < 1$ and $1 < p < \infty$ such that $sp = n$. Then, there exist constants $A_1, A_2, C > 0$ such that for every $u \in H^{s,p}(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ open with finite measure*

$$\int_{\mathbb{R}^n} \exp\left[\frac{|u(x)|}{A_1 C \|D^s u\|_{L^p(\mathbb{R}^n)}}\right]^{p'} dx \leq A_2.$$

Finally, for the remaining case, there is also a *Fractional Morrey inequality* in [99, Theorem. 1.11]. Notice that from this result it can be obtained the fact that $H^{s,p}(\mathbb{R}^n) \subset C^{0,\mu}(\mathbb{R}^n)$ with $0 < \mu \leq s - \frac{n}{p}$, stated in Proposition 3.2.2.

Theorem 3.5.7. *Let $0 < s < 1$ and $1 < p < \infty$ such that $sp > n$. Then, there exists a constant $C = C(n, p, s) > 0$ such that for every $u \in H^{s,p}(\mathbb{R}^n)$*

$$|u(x) - u(y)| \leq C|x - y|^{s-\frac{n}{p}} \|D^s u\|_{L^p(\mathbb{R}^n)}.$$

Putting together all this results the following embedding theorem can be written, from which, in particular, can be obtained a fractional Poincaré inequality.

Theorem 3.5.8. *Set $0 < s < 1$ and $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then there exists $C = C(|\Omega|, n, p, s) > 0$ such that*

$$\|u\|_{L^q(\Omega)} \leq C\|D^s u\|_{L^p(\mathbb{R}^n)}$$

for all $u \in H^{s,p}(\mathbb{R}^n)$, and any q satisfying

$$\begin{cases} q \in [1, p_s^*] & \text{if } sp < n, \\ q \in [1, \infty) & \text{if } sp = n, \\ q \in [1, \infty] & \text{if } sp > n. \end{cases}$$

Proof. The case $sp < n$ is an immediate consequence of Theorem 3.5.3 ([99, Th. 1.8]), where the continuous embedding of $H^{s,p}(\mathbb{R}^n)$ in $L^{p_s^*}(\mathbb{R}^n)$ is shown. Case $sp = n$ is a consequence of Theorem 3.5.6 ([99, Th. 1.10]), where it is proved in this context the version of Trudinger's inequality, which implies the embedding of $H^{s,p}(\mathbb{R}^n)$ in $L^q_{\text{loc}}(\mathbb{R}^n)$ for all $q \in [1, \infty)$. Finally, the case $sp > n$ is a consequence of Proposition 3.2.2 *d)*. □

It is interesting to trace the dependence of the embedding constant on s. Thus, we provide an additional proof of the fractional Poincaré inequality for functions in $H_0^{s,p}(\Omega)$, which uses some ideas of [63, Lemma 7.12], and that will be applied in Chapter 5. Notice that such constant does not depend on p either.

Theorem 3.5.9. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with a Lipschitz boundary. Then there exists $C = C(n, \Omega)$ such that for all $0 < s < 1$, $1 < p < \infty$ and $u \in H_0^{s,p}(\Omega)$,*

$$\|u\|_{L^p(\Omega)} \leq \frac{C}{s} \|D^s u\|_{L^p(\mathbb{R}^n)} .$$

Proof. By density, it is enough to prove the inequality for $u \in C_c^\infty(\Omega)$. Let $R \in \mathbb{R}$, to be specified later, such that

$$R \geq 1, \qquad \Omega \subset B(0, R). \tag{3.43}$$

Define $\Omega_1 := B(0, 2R)$.

Fix $x \in \Omega$. By Theorem 3.5.1 and Lemma 3.1.9,

$$|u(x)| \leq C(n) \left[\int_{\Omega_1} \frac{|D^s u(y)|}{|x-y|^{n-s}}\, dy + \int_{\Omega_1^c} \frac{|D^s u(y)|}{|x-y|^{n-s}}\, dy \right]. \tag{3.44}$$

Now $\Omega_1 \subset B(x, 3R)$, so

$$\int_{\Omega_1} \frac{1}{|x-y|^{n-s}}\, dy \leq \int_{B(x,3R)} \frac{1}{|x-y|^{n-s}}\, dy = \frac{\sigma_{n-1}}{s} (3R)^s \leq C(n) \frac{1}{s} R. \tag{3.45}$$

Similarly, $\Omega \subset B(y, 3R)$ for every $y \in \Omega_1$, so

$$\int_{\Omega} \frac{1}{|x-y|^{n-s}}\, dx \leq C(n) \frac{1}{s} R. \tag{3.46}$$

By (3.45) and Hölder's inequality,

$$\int_{\Omega_1} \frac{|D^s u(y)|}{|x-y|^{n-s}}\, dy \leq \left[C(n) \frac{1}{s} R \right]^{\frac{1}{p'}} \left(\int_{\Omega_1} \frac{|D^s u(y)|^p}{|x-y|^{n-s}}\, dy \right)^{\frac{1}{p}} .$$

Therefore, using (3.46), we find

$$\begin{aligned} &\left[\int_{\Omega} \left(\int_{\Omega_1} \frac{|D^s u(y)|}{|x-y|^{n-s}}\, dy \right)^p dx \right]^{\frac{1}{p}} \leq \\ &\left[C(n) \frac{1}{s} R \right]^{\frac{1}{p'}} \left(\int_{\Omega_1} |D^s u(y)|^p \int_{\Omega} \frac{1}{|x-y|^{n-s}}\, dx\, dy \right)^{\frac{1}{p}} \leq C(n) \frac{1}{s} R \|D^s u\|_p . \end{aligned} \tag{3.47}$$

Now, for any $y \in \Omega_1^c$, by Lemma 3.1.9,

$$|D^s u(y)| \leq C(n) \int \frac{|u(y) - u(z)|}{|y - z|^{n+s}} \, dz = C(n) \int_\Omega \frac{|u(z)|}{|y - z|^{n+s}} \, dz. \tag{3.48}$$

When $z \in \Omega$ we have

$$|y| \leq |y - z| + |z| \leq |y - z| + R \leq |y - z| + \frac{1}{2}|y|,$$

so $\frac{1}{2}|y| \leq |y - z|$ and, hence,

$$\frac{1}{|y - z|^{n+s}} \leq \left(\frac{2}{|y|}\right)^{n+s} \leq C(n) \frac{1}{|y|^{n+s}}. \tag{3.49}$$

Similarly, for each $x \in \Omega$ we have

$$\frac{1}{|x - y|^{n-s}} \leq C(n) \frac{1}{|y|^{n-s}}. \tag{3.50}$$

Using (3.49) we find that

$$\int_\Omega \frac{|u(z)|}{|y - z|^{n+s}} \, dz \leq C(n) \frac{1}{|y|^{n+s}} \|u\|_{L^1(\Omega)} \leq C(n) |\Omega|^{\frac{1}{p'}} \frac{1}{|y|^{n+s}} \|u\|_{L^p(\Omega)},$$

whence we infer from (3.48) that

$$|D^s u(y)| \leq C(n) |\Omega|^{\frac{1}{p'}} \frac{1}{|y|^{n+s}} \|u\|_{L^p(\Omega)}. \tag{3.51}$$

Thus, using (3.50) as well,

$$\begin{aligned}
\int_{\Omega_1^c} \frac{|D^s u(y)|}{|x - y|^{n-s}} \, dy &\leq C(n) |\Omega|^{\frac{1}{p'}} \|u\|_{L^p(\Omega)} \int_{\Omega_1^c} \frac{1}{|y|^{n+s}} \frac{1}{|y|^{n-s}} \, dy \\
&= C(n) |\Omega|^{\frac{1}{p'}} R^{-n} \|u\|_{L^p(\Omega)}.
\end{aligned}$$

This last inequality, combined with (3.44) and (3.47), implies by the triangular inequality that

$$\begin{aligned}
\|u\|_{L^p(\Omega)} &\leq C(n) \frac{1}{s} R \|D^s u\|_p + C_1(n) |\Omega|^{\frac{2}{p'}} R^{-n} \|u\|_{L^p(\Omega)} \\
&\leq C(n) \frac{1}{s} R \|D^s u\|_p + C_1(n) \max\{1, |\Omega|^2\} R^{-n} \|u\|_{L^p(\Omega)}.
\end{aligned}$$

Finally, we choose R such that, in addition to (3.43), satisfies

$$C_1(n)\max\{1, |\Omega|^2\}R^{-n} \le \frac{1}{2},$$

, so that R depends on n and Ω. We obtain that

$$\frac{1}{2}\left\|u\right\|_{L^p(\Omega)} \le C(n)\frac{1}{s}R\left\|D^s u\right\|_{L^p(\mathbb{R}^n)}$$

and concludes the proof. □

We will use the following immediate consequence of Theorem 3.5.9.

Corollary 3.5.10. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $0 < s_0 < 1$. Then there exists $C = C(n, \Omega, s_0)$ such that for all $s_0 < s < 1$, $1 < p < \infty$ and $u \in H_0^{s,p}(\Omega)$,*

$$\left\|u\right\|_{L^p(\Omega)} \le C\left\|D^s u\right\|_p.$$

Following this spirit we also add the continuous embedding of $W^{1,p}(\mathbb{R}^n)$ into $H^{s,p}(\mathbb{R}^n)$ which is already known (see [2, Ch. 7]). Nevertheless, in the next result, we prove it again in order to give an explicit dependence of the embedding constant with respect to s.

Proposition 3.5.11. *Let $1 \le p < \infty$. Then, there exists a constant $C = C(n, p) > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^n)$ and $0 < s < 1$,*

$$\|D^s u\|_p \le \frac{C}{s}\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

Proof. By density, it is enough to prove the inequality for $u \in C_c^\infty(\mathbb{R}^n)$. For all $x \in \mathbb{R}^n$,

$$|D^s u(x)| \le c_{n,s}\left(A(x) + B(x)\right) \tag{3.52}$$

with

$$A(x) := \int_{B(x,1)} \frac{|u(x) - u(y)|}{|x-y|^{n+s}}\, dy, \qquad B(x) := \int_{B(x,1)^c} \frac{|u(x) - u(y)|}{|x-y|^{n+s}}\, dy,$$

so

$$\left\|D^s u\right\|_p \le c_{n,s}\left(\left\|A\right\|_p + \left\|B\right\|_p\right).$$

Note that

$$A(x) = \int_{B(0,1)} \frac{|u(x+h) - u(x)|}{|h|^{n+s}} dh, \qquad B(x) = \int_{B(0,1)^c} \frac{|u(x+h) - u(x)|}{|h|^{n+s}} dh.$$

Applying Minkowski's integral inequality (see, e.g., [110, App. A.1]) we obtain

$$\|A\|_p \leq \int_{B(0,1)} \left(\int \frac{|u(x+h)-u(x)|^p}{|h|^{(n+s)p}}\, dx \right)^{\frac{1}{p}} dh.$$

Now, for all $h \in B(0,1) \setminus \{0\}$,

$$\begin{aligned} \left(\int \frac{|u(x+h)-u(x)|^p}{|h|^{(n+s)p}}\, dx \right)^{\frac{1}{p}} &= \frac{1}{|h|^{n+s}} \left(\int |u(x+h)-u(x)|^p\, dx \right)^{\frac{1}{p}} \\ &\leq \frac{1}{|h|^{n+s-1}} \|Du\|_p, \end{aligned}$$

thanks to a classic inequality (see, e.g., [32, Prop. 9.3] and notice that it is still valid for $p=1$). Therefore,

$$\|A\|_p \leq \|Du\|_p \int_{B(0,1)} \frac{1}{|h|^{n+s-1}} = \frac{\sigma_{n-1}}{1-s} \|Du\|_p, \tag{3.53}$$

where σ_{n-1} is the area of the unit sphere of $\mathbb{R}^n$.

As for B, we first notice that for all $x \in \mathbb{R}^n$, by Hölder's inequality

$$\begin{aligned} B(x) &\leq \int_{B(0,1)^c} \frac{|u(x+h)|}{|h|^{n+s}} dh + \int_{B(0,1)^c} \frac{|u(x)|}{|h|^{n+s}} dh \\ &\leq \left(\int_{B(0,1)^c} \frac{|u(x+h)|^p}{|h|^{n+s}} dh \right)^{\frac{1}{p}} \left(\int_{B(0,1)^c} \frac{1}{|h|^{n+s}} dh \right)^{\frac{1}{p'}} \\ &\quad + |u(x)| \int_{B(0,1)^c} \frac{1}{|h|^{n+s}} dh \\ &= \left(\int_{B(0,1)^c} \frac{|u(x+h)|^p}{|h|^{n+s}} dh \right)^{\frac{1}{p}} \left(\frac{\sigma_{n-1}}{s} \right)^{\frac{1}{p'}} + |u(x)| \frac{\sigma_{n-1}}{s}, \end{aligned}$$

so, by Fubini's theorem,

$$\begin{aligned} \|B\|_p &\leq \left(\frac{\sigma_{n-1}}{s} \right)^{\frac{1}{p'}} \left(\int \int_{B(0,1)^c} \frac{|u(x+h)|^p}{|h|^{n+s}} dh\, dx \right)^{\frac{1}{p}} + \frac{\sigma_{n-1}}{s} \|u\|_p \\ &= 2 \frac{\sigma_{n-1}}{s} \|u\|_p. \end{aligned} \tag{3.54}$$

Putting together (3.52), (3.53) and (3.54), we obtain

$$\|D^s u\|_p \leq c_{n,s} \sigma_{n-1} \left(\frac{1}{1-s} \|Du\|_p + \frac{2}{s} \|u\|_p \right)$$

and, thanks to Lemma 3.1.9, the proof is finished. □

3.5.2 Fractional mean value theorem and compact embeddings

In this subsection we provide an alternative proof of the compactness theorem (based on Frêchet-Kolmogorov theorem instead of Arcoli-Aszelá theorem as in [100]). So as to do so we also obtain a sort of fractional mean value theorem which was lacking in the literature for $p > 1$ (case $p = 1$ was already proved in [37, Proposition 3.13]). First, we introduce the following technical lemma.

Lemma 3.5.12. *There exists a constant $C > 0$, such that for every $s \in (0, 1)$ we have*

$$\int \left| \frac{w}{|w|^{n+1-s}} - \frac{w - e_1}{|w - e_1|^{n+1-s}} \right| dw \leq \frac{C}{s(1-s)}.$$

where e_1 is the first vector of the canonical basis of $\mathbb{R}^n$.

Proof. On the one hand, we have that

$$\int_{B(0,2)} \left| \frac{w}{|w|^{n+1-s}} - \frac{w - e_1}{|w - e_1|^{n+1-s}} \right| dw \leq C \int_{B(0,2)} \frac{1}{|w|^{n-s}} dw \leq C \frac{2^s}{s} \leq \frac{C}{s}.$$

On the other hand, for a fixed $w \in B(0, 2)^c$,

$$\left| \frac{w}{|w|^{n+1-s}} - \frac{w - e_1}{|w - e_1|^{n+1-s}} \right| = \left| \int_0^1 \frac{d}{dt} \frac{w - te_1}{|w - te_1|^{n+1-s}} dt \right| =$$
$$\left| \int_0^1 (n + 1 - s) \frac{[(w - te_1) \cdot e_1](w - te_1)}{|w - te_1|^{n+3-s}} - \frac{e_1}{|w - te_1|^{n+1-s}} dt \right| \leq$$
$$C \int_0^1 \frac{1}{|w - te_1|^{n+1-s}} dt.$$

Now, for $w \in B(0, 2)^c$ and $t \in [0, 1]$ we have

$$|w - te_1| \geq |w| - t \geq |w| - 1 \geq \frac{1}{2}|w|,$$

so

$$\int_0^1 \frac{1}{|w - te_1|^{n+1-s}} dt \leq 2^{n+1-s} \frac{1}{|w|^{n+1-s}} \leq 2^{n+1} \frac{1}{|w|^{n+1-s}}.$$

By integration, we obtain that

$$\int_{B(0,2)^c} \left| \frac{w}{|w|^{n+1-s}} - \frac{w - e_1}{|w - e_1|^{n+1-s}} \right| dw \leq C \int_{B(0,2)^c} \frac{1}{|w|^{n+1-s}} dw \leq C \frac{2^{-1+s}}{1-s}$$
$$\leq \frac{C}{1-s}.$$

This yields the result. $\square$

As it was aforementioned, next result could be considered as a sort of fractional mean value theorem. We prove it for the general case $1 \leq p < \infty$. Actually, it is the fractional version of [32, Proposition 9.3. (iii)].

Proposition 3.5.13. *Let $0 < s < 1$, $1 \leq p < \infty$ and $u \in H^{s,p}(\mathbb{R}^n)$. Then there exists $C > 0$, independent of s, such that*

$$\left(\int |u(x+h) - u(x)|^p dx\right)^{\frac{1}{p}} \leq \frac{C}{s(1-s)} |h|^s \|D^s u\|_{L^p(\mathbb{R}^n)},$$

for every $h \in \mathbb{R}^n$.

Proof. By a standard density argument, it is enough to prove the result for $u \in C_c^\infty(\mathbb{R}^n)$. Let us fix $h \in \mathbb{R}^n$. By Theorem 3.5.1,

$$\begin{aligned} |u(x+h) - u(x)| &= |c_{n,-s}| \left| \int \left(\frac{z}{|z|^{n+1-s}} - \frac{z-h}{|z-h|^{n+1-s}} \right) \cdot D^s u(x+z) dz \right| \\ &\leq |c_{n,-s}| \int \left| \frac{z}{|z|^{n+1-s}} - \frac{z-h}{|z-h|^{n+1-s}} \right| |D^s u(x+z)| dz. \end{aligned} \tag{3.55}$$

Let us take $R \in SO(n)$ (the set of proper rotations in $\mathbb{R}^n$) such that $R^T h = |h| e_1$, where e_1 is the first vector of the canonical basis. Then, making the change of variables $z = |h| R w$, applying Hölder's inequality and using Lemma 3.5.12, we arrive at

$$\begin{aligned} &\int \left| \frac{z}{|z|^{n+1-s}} - \frac{z-h}{|z-h|^{n+1-s}} \right| |D^s u(x+z)| dz \\ =&|h|^s \int \left| \frac{w}{|w|^{n+1-s}} - \frac{w-e_1}{|w-e_1|^{n+1-s}} \right| |D^s u(x+|h|Rw)| dw \\ \leq&|h|^s \left(\frac{C}{s(1-s)} \right)^{\frac{1}{p'}} \left(\int \left| \frac{w}{|w|^{n+1-s}} - \frac{w-e_1}{|w-e_1|^{n+1-s}} \right| |D^s u(x+|h|Rw)|^p dw \right)^{\frac{1}{p}}. \end{aligned}$$

Now, we raise to the p in (3.55), use the previous estimate, integrate, use

Fubini's theorem and apply Lemmas 3.5.12 and 3.1.9 to obtain

$$\begin{aligned}
&\int |u(x+h)-u(x)|^p dx \\
&\leq |c_{n,-s}|^p |h|^{sp} \left(\frac{C}{s(1-s)}\right)^{\frac{p}{p'}} \cdot \\
&\qquad \int \left| \frac{w}{|w|^{n+1-s}} - \frac{w-e_1}{|w-e_1|^{n+1-s}} \right| \int |D^s u(x+|h|Rw)|^p \, dx dw \\
&= |c_{n,-s}|^p |h|^{sp} \left(\frac{C}{s(1-s)}\right)^{\frac{p}{p'}} \|D^s u\|^p_{L^p(\mathbb{R}^n)} \int \left| \frac{w}{|w|^{n+1-s}} - \frac{w-e_1}{|w-e_1|^{n+1-s}} \right| dw \\
&\leq |c_{n,-s}|^p |h|^{sp} \left(\frac{C}{s(1-s)}\right)^{p} \|D^s u\|^p_{L^p(\mathbb{R}^n)}, \\
&\leq |h|^{sp} \left(\frac{C}{s(1-s)}\right)^{p} \|D^s u\|^p_{L^p(\mathbb{R}^n)},
\end{aligned}$$

as desired. □

Next is the compact embedding of $H^{s,p}_g(\Omega)$ into $L^q(\mathbb{R}^n)$ (notice that in [100, Th. 2.2] it was not known yet the density result $H^{s,p}_0(\Omega) = \overline{C^\infty_c(\Omega)}^{\|\cdot\|_{H^{s,p}}}$; the formulation is adapted from [22, Th. 2.3]). In what follows we recall that $p^*_s = \frac{pn}{n-sp}$, and $\rightharpoonup$ denotes weak convergence.

Theorem 3.5.14. *Set $0 < s < 1$ and $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $g \in H^{s,p}(\mathbb{R}^n)$. Then for any sequence $\{u_j\}_{j\in\mathbb{N}} \subset H^{s,p}_g(\Omega)$ such that*

$$u_j \rightharpoonup u \quad \text{in } H^{s,p}(\mathbb{R}^n),$$

for some $u \in H^{s,p}(\mathbb{R}^n)$, one has $u \in H^{s,p}_g(\Omega)$ and

a) $u_j - g \to u - g$ in $L^q(\mathbb{R}^n)$ for every q satisfying

$$\begin{cases} q \in [1, p^*_s) & \text{if } sp < n, \\ q \in [1, \infty) & \text{if } sp = n, \\ q \in [1, \infty] & \text{if } sp > n, \end{cases}$$

b) $u_j \to u$ in $L^q(\mathbb{R}^n)$ for every q satisfying

$$\begin{cases} q \in [p, p^*_s) & \text{if } sp < n, \\ q \in [p, \infty) & \text{if } sp = n, \\ q \in [p, \infty] & \text{if } sp > n. \end{cases}$$

Proof. Here we will focus on proving the case $sp < n$ using Fréchet-Kolmogorov theorem. Case $sp = n$ follows from the former having in mind Proposition 3.2.2, part *c)* or else, part *f)*. Finally, the case $sp > n$ is a consequence of Proposition 3.2.2 *d)* and the compact embedding of $C^{0,\mu}(\bar{\Omega})$ into $C(\bar{\Omega})$.

Thus, let $sp < n$. So as to prove *a)* we assume without loss of generality that $g = 0$. Then, since $\{u_j\}_{j\in\mathbb{N}}$ is a weakly converging sequence in $H^{s,p}(\Omega)$ there exists $C > 0$ such that $\|u_j\|_{H^{s,p}(\mathbb{R}^n)} < C$ for every $j \in \mathbb{N}$. We have to check that, in order to apply Fréchet-Kolmogorov theorem [32, Theorem 4.26],

$$\lim_{|h|\to 0} \|\tau_h u_j - u_j\|_{L^q(\mathbb{R}^n)} = 0 \quad \text{uniformly in } j \in \mathbb{N}, \tag{3.56}$$

where $\tau_h u_j(\cdot) = u_j(\cdot - h)$. By Proposition 3.5.13 we have that there exists $C > 0$ such that

$$\|\tau_h u_j - u_j\|_{L^p(\mathbb{R}^n)} \leq \frac{C}{s(1-s)} |h|^s \|D^s u_j\|_{L^p(\mathbb{R}^n)}. \tag{3.57}$$

Next, considering $p \leq q < p_s^*$, we can write

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p_s^*} \quad \text{for some } \alpha \in (0,1].$$

Let $\tilde{C}$ denote a constant whose value may vary through the different steps, using the interpolation inequality, (3.57) and triangular inequality we obtain

$$\begin{aligned}
\|\tau_h u_j - u_j\|_{L^q(\mathbb{R}^n)} &\leq \|\tau_h u_j - u_j\|^{\alpha}_{L^p(\mathbb{R}^n)} \|\tau_h u_j - u_j\|^{1-\alpha}_{L^{p_s^*}(\mathbb{R}^n)} \\
&\leq \left(\frac{C}{s(1-s)}|h|^s\right)^{\alpha} \|D^s u_j\|^{\alpha}_{L^p(\mathbb{R}^n)} \left(2\|u_j\|_{L^{p_s^*}(\mathbb{R}^n)}\right)^{1-\alpha} \\
&\leq \tilde{C}|h|^{s\alpha} \|D^s u_j\|_{L^p(\mathbb{R}^n)} \leq M\tilde{C}|h|^{s\alpha}
\end{aligned}$$

where we have used Theorem 3.5.9 and the fact that $\|u_j\|_{H^{s,p}(\mathbb{R}^n)} < C$. Thus, (3.56) holds. As a result, Fréchet-Kolmogorov theorem leads to the compact embedding. Notice that since u_j are assumed to have compact support in Ω, there exists $\bar{C} > 0$ such that $\|\tau_h u_j - u_j\|_{L^r(\mathbb{R}^n)} \leq \bar{C}\|\tau_h u_j - u_j\|_{L^q(\mathbb{R}^n)}$ for every $r \in [1, q]$. This proves *a)*. On the other hand, *b)* is obtained when $g \in H^{s,p}(\mathbb{R}^n)$ does not have compact support. □

3.6 Examples of functions in $H^{s,p}(\mathbb{R}^n)$

One of the motivations of this study is to propose an existence theory for variational principles on nonlinear fractional PDE formulated in spaces wider than classical Sobolev spaces. As a consequence of Proposition 3.2.2 *f)*, classical Sobolev spaces are continuously embedded in $H^{s,p}$ spaces. Further, we are interested in functions that belong to $H^{s,p}$ but not to $W^{1,p}$. Necessarily, those functions must exhibit some type of singularity. We focus on two important singularities in solid mechanics: discontinuities along hypersurfaces and at a single point. The later corresponds with the paradigmatic case of cavitation. For simplicity, we study as a model for singularities along hypersurfaces a function whose first component is the characteristic function χ_Q of the unit cube Q, while the other components are C_c^∞ functions. As a model for singularity at a point, we study a radial function of compact support exhibiting one cavity at the origin. In both examples the functions have compact support: this simplifies the analysis since it avoids the issue of the integrability at infinity, and, hence, allows us to focus solely on the singularity.

We start with the case of singularity along a hypersurface. There is an extensive literature on when the characteristic function of a set (especially, of an open bounded Lipschitz set) belongs to a functional space of fractional regularity (see, e.g., [58, 79, 97, 101, 111]). We exploit those results to give a quick proof of the following lemma.

Lemma 3.6.1. *Set $0 < s < 1$ and $1 < p < \infty$. Let $Q = (0,1)^n$ and $\varphi_2, \dots, \varphi_n \in C_c^\infty(\mathbb{R}^n)$. Define $u = (\chi_Q, \varphi_2, \dots, \varphi_n)$. Then*

$$u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \ \ \text{if} \ \ p < \frac{1}{s}, \qquad \text{and} \qquad u \notin H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \ \ \text{if} \ \ p > \frac{1}{s}.$$

Proof. As $C_c^\infty(\mathbb{R}^n) \subset H^{s,p}(\mathbb{R}^n)$ (we will show this in Lemma 3.3.1), we have that $u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ if and only if $\chi_Q \in H^{s,p}(\mathbb{R}^n)$.

The fractional Sobolev space $W^{s,p}$ coincides with the Triebel–Lizorkin space $F^s_{p,p}$ and with the Besov space $B^s_{p,p}$ (see, e.g., [111, Sect. 2.3.5] or [97, Prop. 2.1.2]). This result together with [97, Lemma 4.6.3.2] shows that $\chi_Q \in W^{s,p}$ if and only if $sp < 1$. Proposition 3.2.2 *f)* concludes the proof. □

For the case of cavitation, the result is the following.

Lemma 3.6.2. *Set $0 < s < 1$ and $1 < p < \infty$. Let $\varphi \in C_c^\infty([0,\infty))$ be such that $\varphi(0) > 0$, and $u(x) = \frac{x}{|x|}\varphi(|x|)$. Then*

$$u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \ \ \text{if} \ \ p < \frac{n}{s}, \qquad \text{and} \qquad u \notin H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \ \ \text{if} \ \ p > \frac{n}{s}.$$

Proof. It is well known that $u \in W^{1,q}(\mathbb{R}^n, \mathbb{R}^n)$ whenever $1 < q < n$ (see, e.g., [11, Lemma 4.1]), and therefore $u \in H^{t,q}(\mathbb{R}^n, \mathbb{R}^n)$ for any $0 < t < 1$ and $1 < q < n$. Applying now Proposition 3.2.2 *c)*, we have that $u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$

for any $s \in (0,t)$ and $p \in [q, \frac{nq}{n-(t-s)q}]$. Now we observe that the set of $(s,p) \in \mathbb{R}^2$ such that there exist $q \in (1,n)$ and $t \in (0,1)$ for which $s \in (0,t)$ and $p \in [q, \frac{nq}{n-(t-s)q}]$ is precisely the set of (s,p) such that $s \in (0,1)$ and $p \in (1, \frac{n}{s})$. Therefore, $u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ if $p < \frac{n}{s}$.

On the other hand, when $p > \frac{n}{s}$, by Proposition 3.2.2 *d)*, $H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ functions are continuous. Since u is discontinuous, $u \notin H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ if $p > \frac{n}{s}$. $\square$

Chapter 4. Existence of minimizers of vector fractional functionals under polyconvexity

In this investigation we deepen in the existence issue for vector variational problems involving the s-fractional gradient, as well as the PDE derived from those as equilibrium conditions. In particular, this chapter focuses on the results obtained in [21]. Thus, we consider the more difficult vectorial case under conditions weaker than convexity. To be precise, we establish the existence of minimizers in $H^{s,p}$ under the polyconvexity assumption of the integrand. A key ingredient in this process is the fractional Piola identity

$$\operatorname{Div}^s \operatorname{cof} D^s u = 0$$

(where Div^s means the s-divergence by rows). We believe that the fractional Piola identity is a result of interest in itself. On the one hand, it may serve to show analogous versions in the fractional or nonlocal situations of classical results in whose proof the Piola identity is invoked, as for instance, the change of variables formula for surface integrals. On the other hand, it may also be useful in other fractional or nonlocal models in different contexts, such as fluid mechanics [50]. Furthermore, an extension to a nonlocal Piola identity for nonlocal gradients defined on bounded domains is easy from the proof we provide here in the fractional framework.

The goal of this chapter is to prove the existence of minimizers of fractional vector functionals, following a process inspired by that of J. Ball's in classical hyperelasticity [10]. In other words, based on the direct method of Calculus of Variations, we need to check the coercivity (given by growing conditions and compact embedding results) and the lower semi-continuity properties. With respect to the later, convexity might be a too strict condition in hyperlelasticity, so the proper notion in this framework used by J. Ball was polyconvexity (see, e.g, [10, 39]).

Definition 4.0.1. *Let τ be the number of submatrices of an $n \times n$ matrix. We fix a function $\vec{\mu} : \mathbb{R}^{n\times n} \to \mathbb{R}^\tau$ such that $\vec{\mu}(F)$ is the collection of all minors of an $F \in \mathbb{R}^{n\times n}$ in a given order. A function $W_0 : \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ is polyconvex if there exists a convex $\Phi : \mathbb{R}^\tau \to \mathbb{R} \cup \{\infty\}$ such that*

$$W_0(F) = \Phi(\vec{\mu}(F))$$

for all $F \in \mathbb{R}^{n\times n}$.

Polyconvexity is a central notion in Calculus of Variations, with essential implications in the existence and stability of solutions in solid mechanics, and particularly in elasticity [10, 39] . In order to obtain our results, we follow the usual steps as for classical polyconvex variational problems, namely, we show that the determinant (or any minor) of the fractional gradient matrix $D^s u$ is continuous with respect to weak convergence in $H^{s,p}$.

Thus, the scheme in the classical case was the following. Assuming polyconvexity and proving the weak convergence of the determinant of the gradient would provide the weak lower semi-continuity of the functional. In such process, for a function $u \in C^2(\mathbb{R}^n)$ the Piola Identity

$$\operatorname{Div} \operatorname{cof} Du = 0$$

is a key ingredient (easily proved in the classical case through the Schwartz Theorem of the symmetry of second derivatives), since it allows the determinant of the gradient to be written as a divergence,

$$\det Du = \operatorname{div}(u(\operatorname{cof} Du)).$$

This property is useful as it allows us to use integration by parts, and thus it provides an option to study the weak convergence (for which it is also necessary the compactness given by Rellich-Kondrachov theorem). In particular, for $\{u_j\}_{j\in\mathbb{N}}$ a weakly converging sequence in $W^{1,p}$, $1 < p < \infty$, we have

$$\int \det(Du_j)\varphi = -\frac{1}{n} \int u_j \cdot D\varphi \operatorname{cof} Du_j.$$

Now, the weak convergence of $\det(Du_j)$ is a consequence of the weak convergence obtained for $\operatorname{cof} Du_j$ through an induction process and the strong convergence in L^p of u_j (Rellich-Kondrachov theorem).

This is the scheme we would like to follow in the fractional framework. However some difficulties arise, as the fractional version of the Leibniz formula Lemma 3.3.6 which makes it more difficult to prove a fractional version of the Piola Identity or would affect the integration by part of the determinant of the fractional gradient.

4.1 Fractional Piola Identity

In this section we introduce a fractional version of the Piola Identity. This is the main step in order to prove the existence of solutions for our fractional energy, since it will allow us to prove the weak continuity in $H^{s,p}$ of the determinant of the s-fractional gradient. Recall that the classical Piola identity asserts that, for smooth enough functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ one has $\operatorname{Div} \operatorname{cof} Du = 0$. Of course, cof denotes the cofactor matrix, which satisfies $\operatorname{cof} A\, A^T = (\det A)\, I$ for every $A \in \mathbb{R}^{n\times n}$.

Contrary to the classical case, the proof of the fractional Piola identity is not trivial even for smooth functions. Indeed, the classical proof cannot be reproduced in this case as it relies on Leibniz's rule and symmetry of second derivatives. Notice that Lemma 3.7 prevents all the terms of the second derivatives from being cancelled as happens in the classical case. In the next lines we sketch a possible proof of this identity in order to find out the difficulties. We emphasize that the next argument is formal in order to illustrate the difficulties in proving the fractional Piola identity. The main assumption we make is that all integrals involved are absolutely convergent without the need of the principal value, so that we can apply Fubini's theorem. In this section we will extensively employ the following formulas for the fractional gradient and divergence, obtained from Definition 3.1.2 through odd symmetry,

$$
\begin{aligned}
D^s u(x) &= -c_{n,s} \int \frac{u(y)}{|x-y|^{n+s+1}} \otimes (x-y)\,\mathrm{d}y; \\
\operatorname{div}^s \phi(x) &= -c_{n,s}\, \mathrm{pv}_x \int \frac{\phi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} dy.
\end{aligned}
\tag{4.1}
$$

For simplicity in the calculations, we set $n = 2$, the simplest case. Then, for $u : \mathbb{R}^2 \to \mathbb{R}^2$, with $u_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, its components, the first row of $\operatorname{cof} D^s u(x)$ is

$$
\begin{aligned}
&(D^s_2 u_2(x), D^s_1 u_2(x)) = \\
&- c_{n,s} \left(\int \frac{u_2(y)}{|x-y|^{n+s+1}} (x_2 - y_2)\,\mathrm{d}y,\ - \int \frac{u_2(y)}{|x-y|^{n+s+1}} (x_1 - y_1)\,\mathrm{d}y \right),
\end{aligned}
$$

and, using (4.1), the first component of $\operatorname{Div}^s \operatorname{cof} D^s u(x)$ is

$$
\begin{aligned}
c_{n,s}^2 \int \Bigg[\int & \frac{u_2(z)}{|y-z|^{n+s+1}} (y_2 - z_2) \, \mathrm{d}z \, (x_1 - y_1) \\
& - \int \frac{u_2(z)}{|y-z|^{n+s+1}} (y_1 - z_1) \, \mathrm{d}z \, (x_2 - y_2) \Bigg] \frac{1}{|x-y|^{n+s+1}} \mathrm{d}y \\
= c_{n,s}^2 \int u_2(z) & \int \frac{\det(x-y, y-z)}{|x-y|^{n+s+1} |y-z|^{n+s+1}} \, \mathrm{d}y \, \mathrm{d}z.
\end{aligned}
$$

Then, the problem is solved is

$$
\mathrm{pv} \int \frac{\det(x-y, y-z)}{|x-y|^{n+s+1} |y-z|^{n+s+1}} \, \mathrm{d}y = 0 \tag{4.2}
$$

Notice that the integrals in (4.2) are not defined as Lebesgue integrals. The real proof will consist in making these calculations rigorous for arbitrary dimension n. The underlying reason of why the fractional Piola identity is true is that $\det D^s u$ is a sort of null Lagrangian in the sense that, for any $n \geq 2$, the integral

$$
\mathrm{pv} \int \frac{\det(x-a_1, \ldots, x-a_n)}{|x-a_1|^{n+s+1} \cdots |x-a_n|^{n+s+1}} \, dx
$$

is zero. This is a consequence of the fact that the determinant is an alternating multilinear form, as well as that $\det Du$ is a classical null Lagrangian. However, as we will see in Lemma 4.1, the previous integral is not defined as a proper integral but as a principal value centered at points $a_1, \ldots, a_n$, and this will cause technical difficulties in the proof.

We start by reviewing a version of the change of variables formula for surface integrals (see, e.g., [86, Prop. 2.7]). Let Γ be an oriented $(n-1)$-dimensional manifold with continuous unit normal field ν. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be affine and injective, with corresponding linear map $\vec{T}$. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be smooth. Then

$$
\int_\Gamma g(Tx) \cdot \operatorname{cof} \vec{T} \nu(x) \, dS(x) = \int_{T(\Gamma)} g(x) \cdot \frac{\operatorname{cof} \vec{T} \nu(T^{-1}x)}{|\operatorname{cof} \vec{T} \nu(T^{-1}x)|} \, dS(x),
$$

where dS denotes the surface element. Now assume that T is a symmetry across a hyperplane, so $T^{-1} = T$, $\det \vec{T} = -1$ and $\vec{T}^{-1} = \vec{T} = \vec{T}^T = -\operatorname{cof} \vec{T}$. Therefore,

$$
-\int_\Gamma g(Tx) \cdot \vec{T} \nu(x) \, dS(x) = -\int_{T(\Gamma)} g(x) \cdot \vec{T} \nu(Tx) \, dS(x).
$$

Thus

$$\int_{\Gamma} \vec{T}g(Tx) \cdot \nu(x)\, dS(x) = \int_{T(\Gamma)} \vec{T}g(x) \cdot \nu(Tx)\, dS(x).$$

As this is true for every g, we have that

$$\int_{\Gamma} g(x) \cdot \nu(x)\, dS(x) = \int_{T(\Gamma)} g(Tx) \cdot \nu(Tx)\, dS(x), \tag{4.3}$$

which is the formula we will use in Lemma 4.1.1.

In this and the next sections we will employ the following notation for the submatrices.

Definition 4.1.1. *Let $k \in \mathbb{N}$ be with $1 \leq k \leq n$. Consider indices $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$.*

a) We define $[\cdot]_M = [\cdot]_{M_{i_1,\ldots,i_k;j_1,\ldots,j_k}} : \mathbb{R}^{n\times n} \to \mathbb{R}^{k\times k}$ as the map such that $[F]_M$ is the submatrix of $F \in \mathbb{R}^{n\times n}$ formed by the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$.

b) We define $[\cdot]_{\bar{M}} = [\cdot]_{\bar{M}_{i_1,\ldots,i_k;j_1,\ldots,j_k}} : \mathbb{R}^{k\times k} \to \mathbb{R}^{n\times n}$ as the map such that $[F]_{\bar{M}}$ is the matrix whose rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_k$ coincide with those of F, whereas the rest of the entries are zero.

c) We define $[\cdot]_N = [\cdot]_{N_{i_1,\ldots,i_k}} : \mathbb{R}^n \to \mathbb{R}^k$ as the map such that $[v]_N$ is the subvector of $v \in \mathbb{R}^n$ formed by the entries $i_1, \ldots, i_k$.

d) We define $[\cdot]_{\bar{N}} = [\cdot]_{\bar{N}_{i_1,\ldots,i_k}} : \mathbb{R}^k \to \mathbb{R}^n$ as the map such that $[v]_{\bar{N}}$ is the vector whose entries $i_1, \ldots, i_k$ coincide with those of v, whereas the rest of the entries are zero.

e) We define $[\cdot]_{\tilde{N}} = [\cdot]_{\tilde{N}_{i_1,\ldots,i_k}} : \mathbb{R}^n \to \mathbb{R}^n$ as $[\cdot]_{\bar{N}} \circ [\cdot]_N = [[\cdot]_N]_{\bar{N}}$.

To clarify this notation, we have used letters without diacritical marks to denote a submatrix or a subvector whereas the ones with diacritical marks are used for the extended versions (by zeros).

The following formulas for the determinant will be useful. Given $A \in \mathbb{R}^{n\times n}$, we express it as

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix},$$

where $a_1, \dots, a_n \in \mathbb{R}^n$ are its rows. Then $\det A = a_i \cdot (\operatorname{cof} A)_i$ for each $i \in \{1, \dots, n\}$, where $(\operatorname{cof} A)_i$ denotes the i-th row of $\operatorname{cof} A$. Now we realize that if $b \in \mathbb{R}^n$ and

$$A' = \begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ b \\ a_{i+1} \\ \vdots \\ a_n \end{pmatrix},$$

then

$$\det A' = (\operatorname{cof} A)_i \cdot b. \tag{4.4}$$

The following lemma is the rigorous version of (4.2).

Lemma 4.1.1. *Let $k \in \mathbb{N}$ be with $1 \leq k \leq n$. Consider indices $1 \leq j_1 < \cdots < j_k \leq n$ and let $[\cdot]_N = [\cdot]_{N_{j_1,\dots,j_k}}$ be the function of Definition 4.1.1. Then there exists a continuous function $G : [0, \infty) \times (\mathbb{R}^n)^{k-1} \to \mathbb{R}$ such that for any $a_1, \dots, a_k \in \mathbb{R}^n$ and $\epsilon_1, \dots, \epsilon_k > 0$ we have*

$$\left| \int_{\left(\bigcup_{j=1}^k B(a_j,\epsilon_j)\right)^c} \frac{\det([x-a_1]_N, \dots, [x-a_k]_N)}{|x-a_1|^{n+s+1} \cdots |x-a_k|^{n+s+1}} dx \right| \leq \frac{\epsilon_1^{1-s}}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} G(\epsilon_1, a_2 - a_1, \dots, a_k - a_1).$$

Proof. We can assume that the points $a_1, \dots, a_k$ do not lie on an affine manifold of dimension $k-2$, since otherwise $\det([x-a_1]_N, \dots, [x-a_k]_N) = 0$ for all $x \in \mathbb{R}^n$.

Define $h : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ as

$$h(x) = \frac{-1}{(n+s-1)|x|^{n+s-1}} \tag{4.5}$$

and $h_i : \mathbb{R}^n \setminus \{a_i\} \to \mathbb{R}$ as $h_i(x) = h(x - a_i)$, for each $i = 1, \dots, k$. Define $H : \mathbb{R}^n \setminus \{a_1, \dots, a_k\} \to \mathbb{R}^k$ componentwise as $H = (h_1, \dots, h_k)^T$. Then

$$DH(x) = \begin{pmatrix} \nabla h_1(x) \\ \vdots \\ \nabla h_k(x) \end{pmatrix} = \begin{pmatrix} \frac{x-a_1}{|x-a_1|^{n+s+1}} \\ \vdots \\ \frac{x-a_k}{|x-a_k|^{n+s+1}} \end{pmatrix}. \tag{4.6}$$

Call $\vec{\jmath} = (j_1, \dots, j_k)$ and denote by $D_{\vec{\jmath}}H$ the submatrix of DH formed by the columns $j_1, \dots, j_k$. Then, for all $x \in \mathbb{R}^n \setminus \{a_1, \dots, a_k\}$,

$$\det D_{\vec{\jmath}}H(x) = \frac{\det([x-a_1]_N, \dots, [x-a_k]_N)}{|x-a_1|^{n+s+1} \cdots |x-a_k|^{n+s+1}}. \tag{4.7}$$

As $DH \in L^p\big(\big(\bigcup_{j=1}^k B(a_j, \epsilon_j)\big)^c, \mathbb{R}^{n\times n}\big)$ for all $p \in [1, \infty]$, we have $\det D_{\vec{\jmath}}H \in L^1\big(\big(\bigcup_{j=1}^k B(a_j, \epsilon_j)\big)^c\big)$. Therefore,

$$\int_{\left(\bigcup_{j=1}^k B(a_j,\epsilon_j)\right)^c} \det D_{\vec{\jmath}}H = \lim_{R\to\infty} \int_{B(0,R)\setminus\bigcup_{j=1}^k B(a_j,\epsilon_j)} \det D_{\vec{\jmath}}H.$$

As H is smooth outside $\bigcup_{j=1}^k B(a_j, \epsilon_j)$, we have that

$$\det D_{\vec{\jmath}}H = \operatorname{div}[h_1(\operatorname{cof} D_{\vec{\jmath}}H)_1]_{\bar{N}},$$

where $(\operatorname{cof} D_{\vec{\jmath}}H)_1$ indicates the first row of $\operatorname{cof} D_{\vec{\jmath}}H$, and $[\cdot]_{\bar{N}} = [\cdot]_{\bar{N}_{j_1,\dots,j_k}}$ is the function of Definition 4.1.1. Let $R > 0$ be big enough so that $\bigcup_{j=1}^k \bar{B}(a_j, \epsilon_j) \subset B(0, R)$. Then, by the divergence theorem,

$$\int_{B(0,R)\setminus\bigcup_{j=1}^k B(a_j,\epsilon_j)} \det D_{\vec{\jmath}}H =$$
$$-\int_{\partial\bigcup_{j=1}^k B(a_j,\epsilon_j)} [h_1(\operatorname{cof} D_{\vec{\jmath}}H)_1]_{\bar{N}} \cdot \nu_j + \int_{\partial B(0,R)} [h_1(\operatorname{cof} D_{\vec{\jmath}}H)_1]_{\bar{N}} \cdot \nu_R,$$

where $\nu_j(x) = \frac{x-a_j}{\epsilon_j}$ in $\partial B(a_j, \epsilon_j)$ for $j = 1, \dots, k$, and $\nu_R(x) = \frac{x}{R}$ in $\partial B(0, R)$. Having in mind the expressions (4.5) and (4.6), we find that, for some constant $C > 0$,

$$\left| \int_{\partial B(0,R)} [h_1(\operatorname{cof} D_{\vec{\jmath}}H)_1]_{\bar{N}} \cdot \nu_R \right| \le \frac{C}{R^{(n+s)k-1}},$$

which goes to zero as $R \to \infty$. Therefore,

$$\int_{\left(\bigcup_{j=1}^k B(a_j,\epsilon_j)\right)^c} \det D_{\vec{\jmath}}H = -\int_{\partial\bigcup_{j=1}^k B(a_j,\epsilon_j)} [h_1(\operatorname{cof} D_{\vec{\jmath}}H)_1]_{\bar{N}} \cdot \nu_j. \tag{4.8}$$

For each $i = 1, \dots, n$ we set

$$A_i = \partial\left(\bigcup_{j=1}^k B(a_j, \epsilon_j)\right) \cap \partial B(a_i, \epsilon_i).$$

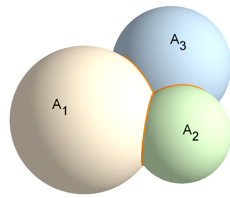


Figure 4.1: Sets A_1, A_2, A_3 in $\mathbb{R}^3$

As a consequence of the inclusion $\partial \bigcup_{j=1}^{k} B(a_j, \epsilon_j) \subset \bigcup_{j=1}^{k} \partial B(a_j, \epsilon_j)$, we have that

$$\partial \bigcup_{j=1}^{k} B(a_j, \epsilon_j) = \bigcup_{j=1}^{k} A_j.$$

Moreover, the $(n-1)$-dimensional area of $A_i \cap A_j$ is zero for $1 \leq i < j \leq k$. Figure 4.1 illustrates this situation when $k = n = 3$.

Next, using (4.4) and (4.6), we have that for $j = 2, \ldots, k$ and $x \in \partial B(a_j, \epsilon_j)$,

$$[h_1(\operatorname{cof} D_{\bar{j}} H)_1]_{\bar{N}} \cdot \nu_j(x) = \frac{\det([x - a_j]_N, [x - a_2]_N, \ldots, [x - a_k]_N)}{|x - a_j|\, |x - a_2|^{n+s+1} \cdots |x - a_k|^{n+s+1}} = 0.$$

As a result, recalling (4.8) and the inclusion $A_j \subset \partial B(a_j, \epsilon_j)$, we have that

$$\int_{\left(\bigcup_{j=1}^{k} B(a_j, \epsilon_j)\right)^c} \det D_{\bar{j}} H \, dx = -\int_{A_1} [h_1(\operatorname{cof} D_{\bar{j}} H)_1]_{\bar{N}} \cdot \nu_1 \, dS. \tag{4.9}$$

Having in mind the expression (4.5), the multilinearity of the determinant and considering (4.4) and (4.6), we have that, for $x \in A_1$,

$$\begin{aligned}
-[h_1(\operatorname{cof} D_{\bar{j}} H)_1]_{\bar{N}} \cdot \nu_1(x) &= \frac{1}{n+s-1} \frac{1}{\epsilon_1^{n+s}} (\operatorname{cof} D_{\bar{j}} H)_1 \cdot [x - a_1]_N \\
&= \frac{1}{n+s-1} \frac{1}{\epsilon_1^{n+s}} \frac{\det([x - a_1]_N, [x - a_2]_N, \ldots, [x - a_k]_N)}{|x - a_2|^{n+s+1} \cdots |x - a_k|^{n+s+1}} \\
&= \frac{1}{n+s-1} \frac{1}{\epsilon_1^{n+s}} \frac{\det([x - a_1]_N, [a_1 - a_2]_N, \ldots, [a_1 - a_k]_N)}{|x - a_2|^{n+s+1} \cdots |x - a_k|^{n+s+1}} \\
&= \frac{1}{n+s-1} \frac{1}{\epsilon_1^{n+s-1}} \frac{([\operatorname{cof}([x - a_1]_N, [a_1 - a_2]_N, \ldots, [a_1 - a_k]_N)]_{\bar{M}})_1}{|x - a_2|^{n+s+1} \cdots |x - a_k|^{n+s+1}} \cdot \nu_1(x),
\end{aligned} \tag{4.10}$$

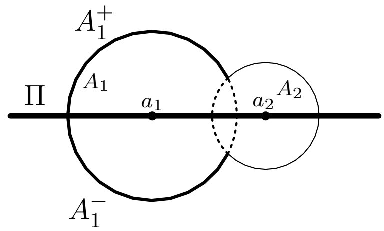


Figure 4.2: Sets A_1, A_2, A_1^+, A_1^- and Π

where $[\cdot]_{\bar{M}} = [\cdot]_{\bar{M}_{i_1,\ldots,i_k;j_1,\ldots,j_k}}$ is the function of Definition 4.1.1.

Let Π_k be the only hyperplane in $\mathbb{R}^k$ such that the points $[a_1]_N, \ldots, [a_k]_N$ belong to Π_k, and consider one of the two unit normals $\vec{n} \in \mathbb{R}^k$ to Π_k. Let $T_k : \mathbb{R}^k \to \mathbb{R}^k$ be the symmetry with respect to Π_k, so that for every $y \in \mathbb{R}^k$,

$$T_k y = y - 2(y - [a_1]_N) \cdot \vec{n}. \tag{4.11}$$

Let $\vec{m} = [\vec{n}]_{\bar{N}}$, and let Π be the affine hyperplane in $\mathbb{R}^n$ with normal $\vec{m}$ passing through a_1. Consider $T : \mathbb{R}^n \to \mathbb{R}^n$ as the symmetry across Π. Then, for all $x \in \mathbb{R}^n$,

$$Tx = x - 2(x - a_1) \cdot \vec{m}. \tag{4.12}$$

Let $a_{k+1}, \ldots, a_n \in \Pi$ be such that the points $a_1, \ldots, a_n$ do not lie in an affine manifold of dimension $n-2$. Define $A_1^{\pm} = \{x \in A_1 : \pm \det(x - a_1, a_1 - a_2, \ldots, a_1 - a_n) > 0\}$. Then $T(A_1^{\pm}) = A_1^{\mp}$, and $A_1^+ \cup A_1^-$ cover A_1 up to a set of zero $(n-1)$-measure; see Figure 4.2. Using the change of variables formula (4.3), we obtain

$$\begin{aligned} &\int_{A_1^-} \frac{([\operatorname{cof}([x-a_1]_N, [a_1-a_2]_N, \ldots, [a_1-a_k]_N)]_{\bar{M}})_1}{|x-a_2|^{n+s+1} \cdots |x-a_k|^{n+s+1}} \cdot \nu_1(x)\, dS(x) \\ &= \int_{A_1^+} \frac{([\operatorname{cof}([Tx-a_1]_N, [a_1-a_2]_N, \ldots, [a_1-a_k]_N)]_{\bar{M}})_1}{|Tx-a_2|^{n+s+1} \cdots |Tx-a_k|^{n+s+1}} \cdot \nu_1(Tx)\, dS(x). \end{aligned} \tag{4.13}$$

Now, thanks to (4.4), for $x \in A_1^+$,

$$\begin{aligned} &([\operatorname{cof}([Tx-a_1]_N, [a_1-a_2]_N, \ldots, [a_1-a_k]_N)]_{\bar{M}})_1 \cdot \nu_1(Tx) \\ &= \frac{1}{\epsilon_1} \det([Tx-a_1]_N, [a_1-a_2]_N, \ldots, [a_1-a_k]_N). \end{aligned} \tag{4.14}$$

Let $\vec{T}_k : \mathbb{R}^k \to \mathbb{R}^k$ be the linear map corresponding to the affine map T_k, and, analogously, $\vec{T} : \mathbb{R}^n \to \mathbb{R}^n$ the linear map corresponding to T. We notice that $\det \vec{T}_k = -1$. Having in mind (4.11) and (4.12), we find that

$$\vec{T}_k y = y - 2y \cdot \vec{n}, \qquad y \in \mathbb{R}^k$$

and

$$\vec{T} x = x - 2x \cdot \vec{m}, \qquad x \in \mathbb{R}^n,$$

from which we deduce that $\vec{T}_k \circ [\cdot]_N = [\cdot]_N \circ \vec{T}$. Thus,

$$\begin{aligned} &\det\left([Tx - a_1]_N, [a_1 - a_2]_N, \dots, [a_1 - a_k]_N\right) \\ &= \det([Tx - Ta_1]_N, [Ta_1 - Ta_2]_N, \dots, [Ta_1 - Ta_k]_N) \\ &= \det([\vec{T}(x - a_1)]_N, [\vec{T}(a_1 - a_2)]_N, \dots, [\vec{T}(a_1 - a_k)]_N) \\ &= \det(\vec{T}_k([x - a_1]_N), \vec{T}_k([a_1 - a_2]_N), \dots, \vec{T}_k([a_1 - a_k]_N)) \\ &= \det \vec{T}_k([x - a_1]_N, [a_1 - a_2]_N, \dots, [a_1 - a_k]_N) \\ &= -\det([x - a_1]_N, [a_1 - a_2]_N, \dots, [a_1 - a_k]_N). \end{aligned} \tag{4.15}$$

Putting together (4.13), (4.14) and (4.15), we obtain that

$$\begin{aligned} &\int_{A_1^-} \frac{\det([x - a_1]_N, [a_1 - a_2]_N, \dots, [a_1 - a_k]_N)}{|x - a_2|^{n+s+1} \cdots |x - a_k|^{n+s+1}} \, dS(x) = \\ &- \int_{A_1^+} \frac{\det([x - a_1]_N, [a_1 - a_2]_N, \dots, [a_1 - a_k]_N)}{|Tx - a_2|^{n+s+1} \cdots |Tx - a_k|^{n+s+1}} \, dS(x). \end{aligned}$$

Consequently, when we define $f : \mathbb{R}^n \setminus \{a_2, \dots, a_k\} \to \mathbb{R}$ as

$$f(y) := \frac{1}{(|y - a_2| \cdots |y - a_k|)^{n+s+1}},$$

we have that

$$\begin{aligned} &\int_{A_1} \frac{\det([x - a_1]_N, [a_1 - a_2]_N, \dots, [a_1 - a_k]_N)}{|x - a_2|^{n+s+1} \cdots |x - a_k|^{n+s+1}} \, dS(x) = \\ &\int_{A_1^+} \det([x - a_1]_N, [a_1 - a_2]_N, \dots, [a_1 - a_k]_N) \, [f(x) - f(Tx)] \, dS(x). \end{aligned} \tag{4.16}$$

For every $x \in A_1^+$, we join x with Tx by a curve γ_x inside A_1, and note that the length of γ_x can be taken to be bounded by $2\pi\varepsilon_1$. Accordingly, let

$\gamma_x : [0,1] \to A_1$ be of class C^1 such that $\gamma_x(0) = x$, $\gamma_x(1) = Tx$ and $|\gamma_x'|$ is constant with $|\gamma_x'| \leq 2\pi\varepsilon_1$. Then

$$\begin{aligned} |f(x) - f(Tx)| = |f(\gamma_x(0)) - f(\gamma_x(1))| &\leq \int_0^1 |\gamma_x'| \, |\nabla f(\gamma_x(t))| \, dt \\ &\leq 2\pi\epsilon_1 \int_0^1 |\nabla f(\gamma_x(t))| \, dt. \end{aligned} \tag{4.17}$$

We calculate

$$|\nabla f(y)| = (n+s+1) \left(|y-a_2| \cdots |y-a_k| \right)^{-n-s-2} \sum_{i=2}^{k} \prod_{\substack{j=2 \\ j\neq i}}^{k} |y-a_j| \, , \tag{4.18}$$

for $y \in \mathbb{R}^n \setminus \{a_2, \dots, a_k\}$.

Now, as $|y - a_j| > \epsilon_j$ for every $y \in A_1$ and $j \in \{2, \dots, k\}$,

$$\begin{aligned} |\nabla f(y)| &\leq \frac{n+s+1}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} \sum_{i=2}^{k} \prod_{\substack{j=2 \\ j\neq i}}^{k} |y-a_j| \\ &\leq \frac{n+s+1}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} \sum_{i=2}^{k} \prod_{\substack{j=2 \\ j\neq i}}^{k} (\epsilon_1 + |a_1 - a_j|), \end{aligned}$$

so with (4.17) we obtain that

$$|f(x) - f(Tx)| \leq 2\pi\epsilon_1 \frac{n+s+1}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} \sum_{i=2}^{k} \prod_{\substack{j=2 \\ j\neq i}}^{k} (\epsilon_1 + |a_1 - a_j|). \tag{4.19}$$

On the other hand, for all $x \in A_1$,

$$\begin{aligned} &|\det([x-a_1]_N, [a_1-a_2]_N, \dots, [a_1-a_k]_N)| \\ &\leq k! \, |x - a_1| \prod_{j=2}^{k} |a_1 - a_j| = k! \, \epsilon_1 \prod_{j=2}^{k} |a_1 - a_j| \, . \end{aligned} \tag{4.20}$$

Putting together (4.7), (4.9), (4.10), (4.16), (4.19) and (4.20), as well as the fact that the $(n-1)$-dimensional area of A_1^+ is bounded by a constant times

ϵ_1^{n-1}, we obtain that, for a constant $C > 0$ depending on n and s,

$$\left| \int_{\left(\bigcup_{j=1}^k B(a_j,\epsilon_j)\right)^c} \frac{\det([x-a_1]_N, [a_1-a_2]_N, \ldots, [a_1-a_k]_N)}{|x-a_1|^{n+s+1}\cdots|x-a_k|^{n+s+1}} dx \right| \leq$$
$$\frac{C\,\epsilon_1^{1-s}}{(\epsilon_2\cdots\epsilon_k)^{n+s+2}} \left(\prod_{j=2}^k |a_1-a_j| \right) \sum_{i=2}^k \prod_{\substack{j=2 \\ j\neq i}}^k (\epsilon_1 + |a_1-a_j|).$$

The existence of the function G of the statement follows. $\square$

We are in a position to prove the fractional Piola Identity. Henceforth, supp denotes the support of a function.

Theorem 4.1.2. *Let $k \in \mathbb{N}$ be with $1 \leq k \leq n$. Consider indices $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$ and the functions*

$$[\cdot]_M = [\cdot]_{M_{i_1,\ldots,i_k;j_1,\ldots,j_k}}, \quad [\cdot]_{\bar{M}} = [\cdot]_{\bar{M}_{i_1,\ldots,i_k;j_1,\ldots,j_k}}$$

of Definition 4.1.1. Let $u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and $s \in (0,1)$. Then

$$\operatorname{Div}^s([\operatorname{cof}[D^s u]_M]_{\bar{M}}) = 0.$$

Proof. Let

$$[\cdot]_N = [\cdot]_{N_{j_1,\ldots,j_k}}, \quad [\cdot]_{\bar{N}} = [\cdot]_{\bar{N}_{j_1,\ldots,j_k}}$$

be the maps of Definition 4.1.1. Naturally, $\operatorname{Div}^s([\operatorname{cof}[D^s u]_M]_{\bar{M}}) = 0$ if and only if

$$\operatorname{div}^s[(\operatorname{cof}[D^s u]_M)_{i_\ell}]_{\bar{N}} = 0, \qquad \ell = 1, \ldots, k.$$

We shall show $\operatorname{div}^s[(\operatorname{cof}[D^s u]_M)_{i_1}]_{\bar{N}} = 0$. The rest of the rows would proceed analogously.

Using (4.1), we have that, for a.e. $x \in \mathbb{R}^n$,

$$\begin{aligned} &\frac{(-1)^{k-1}}{c_{n,s}^k} \operatorname{div}^s[(\operatorname{cof}[D^s u]_M)_{i_1}]_{\bar{N}}(x) = \\ &\frac{(-1)^{k-1}}{c_{n,s}^{k-1}} \operatorname{pv}_x \int \frac{[(\operatorname{cof}[D^s u]_M)_{i_1}]_{\bar{N}}(x')}{|x'-x|^{n+s+1}} \cdot (x'-x)\, dx'. \end{aligned} \tag{4.21}$$

Now, by (4.4) and (4.1), we have that for a.e. $x, x' \in \mathbb{R}^n$,

$$
\begin{aligned}
&\frac{(-1)^{k-1}}{c_{n,s}^{k-1}} \frac{[(\operatorname{cof}[D^s u]_M)_{i_1}]_{\bar{N}}(x')}{|x'-x|^{n+s+1}} \cdot (x'-x) = \frac{(-1)^{k-1}}{c_{n,s}^{k-1}} \frac{(\operatorname{cof}[D^s u]_M)_{i_1}(x')}{|x'-x|^{n+s+1}} \cdot [x'-x]_N \\
&= \frac{(-1)^{k-1}}{c_{n,s}^{k-1}} \frac{\det\left([x'-x]_N, [D^s u_{i_2}(x')]_N, \dots, [D^s u_{i_k}(x')]_N\right)}{|x'-x|^{n+s+1}} \\
&= \det\Bigg(\frac{[x'-x]_N}{|x'-x|^{n+s+1}}, \operatorname{pv}_{x'} \int \frac{u_{i_2}(y_2)[x'-y_2]_N}{|x'-y_2|^{n+s+1}}\, dy_2, \dots, \\
&\qquad\qquad \operatorname{pv}_{x'} \int \frac{u_{i_k}(y_k)[x'-y_k]_N}{|x'-y_k|^{n+s+1}}\, dy_k \Bigg) \\
&= \lim_{\varepsilon_2 \to 0} \cdots \lim_{\varepsilon_k \to 0} f^x_{\varepsilon_2,\dots,\varepsilon_k}(x'),
\end{aligned}
\tag{4.22}
$$

where for each $x \in \mathbb{R}^n$ and $\varepsilon_2, \dots, \varepsilon_k > 0$, we have defined $f^x_{\varepsilon_2,\dots,\varepsilon_k} : \mathbb{R}^n \to \mathbb{R}$ by

$$
\begin{aligned}
f^x_{\varepsilon_2,\dots,\varepsilon_k}(x') := \det\Bigg(&\frac{[x'-x]_N}{|x'-x|^{n+s+1}}, \int_{B(x',\varepsilon_2)^c} \frac{u_{i_2}(y_2)[x'-y_2]_N}{|x'-y_2|^{n+s+1}}\, dy_2, \dots, \\
&\int_{B(x',\varepsilon_k)^c} \frac{u_{i_k}(y_k)[x'-y_k]_N}{|x'-y_k|^{n+s+1}}\, dy_k \Bigg)
\end{aligned}
$$

and we have used the continuity of the determinant. Let $\rho > 0$ be such that $\operatorname{supp} u \subset B(x', \rho)$ for all $x' \in \operatorname{supp} u$, and fix $\ell \in \{2, \dots, k\}$. By odd symmetry, we have that

$$
\begin{aligned}
&\int_{B(x',\varepsilon_j)^c} u_{i_\ell}(y_\ell) \frac{[x'-y_\ell]_N}{|x'-y_\ell|^{n+s+1}} dy_\ell = \int_{B(x',\rho)\setminus B(x',\varepsilon_j)} u_{i_\ell}(y_\ell) \frac{[x'-y_\ell]_N}{|x'-y_\ell|^{n+s+1}} dy_\ell = \\
&\int_{B(x',\rho)\setminus B(x',\varepsilon_j)} \left(u_{i_\ell}(y_\ell) - u_{i_\ell}(x')\right) \frac{[x'-y_\ell]_N}{|x'-y_\ell|^{n+s+1}} dy_\ell,
\end{aligned}
$$

so, using the fact that u is Lipschitz, we have, for some constant $L > 0$, that

$$
\begin{aligned}
\left| \int_{B(x',\varepsilon_j)^c} u_{i_\ell}(y_\ell) \frac{[x'-y_\ell]_N}{|x'-y_\ell|^{n+s+1}} dy_\ell \right| &\le \int_{B(x',\rho)} \frac{|u_{i_\ell}(y_\ell) - u_{i_\ell}(x')|}{|x'-y_\ell|^{n+s}} dy_\ell \\
&\le L \int_{B(x',\rho)} \frac{1}{|x'-y_\ell|^{n+s-1}} dy_\ell \\
&= L \int_{B(0,\rho)} \frac{1}{|y|^{n+s-1}} dy < \infty.
\end{aligned}
$$

This shows that
$$\left|f^x_{\varepsilon_2,\dots,\varepsilon_k}(x')\right| \leq \frac{c}{|x'-x|^{n+s}}$$
for some $c>0$ only depending on u and n. As
$$\int_{B(x,\varepsilon_1)^c} \frac{1}{|x'-x|^{n+s}}dx' < \infty,$$
for any $\varepsilon_1>0$, we can apply dominated convergence to conclude that
$$\int_{B(x,\varepsilon_1)^c} \lim_{\varepsilon_2\to 0}\cdots\lim_{\varepsilon_k\to 0} f^x_{\varepsilon_2,\dots,\varepsilon_k}(x')\,dx' = \lim_{\varepsilon_2\to 0}\cdots\lim_{\varepsilon_k\to 0}\int_{B(x,\varepsilon_1)^c} f^x_{\varepsilon_2,\dots,\varepsilon_k}(x')\,dx'.$$
Recalling (4.21) and (4.22), with this we obtain that
$$\frac{(-1)^{k-1}}{c^k_{n,s}}\operatorname{div}^s[(\operatorname{cof}[D^s u]_M)_{i_1}]_{\bar N}(x) = \lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0}\cdots\lim_{\varepsilon_k\to 0}\int_{B(x,\varepsilon_1)^c} f^x_{\varepsilon_2,\dots,\varepsilon_k}(x')\,dx'. \tag{4.23}$$
Now for every $\varepsilon_1,\dots,\varepsilon_k>0$ we define $D_{\varepsilon_1,\dots,\varepsilon_k} := B(x,\varepsilon_1)\cup\bigcup_{j=2}^k B(y_j,\varepsilon_j)$ and have that, thanks to the multilinearity of the determinant,
$$\begin{aligned}
&\int_{B(x,\varepsilon_1)^c} f^x_{\varepsilon_2,\dots,\varepsilon_k}(x')\,dx'\\
&= \int_{B(x,\varepsilon_1)^c}\int_{B(x',\varepsilon_2)^c}\cdots\int_{B(x',\varepsilon_k)^c}\\
&\qquad \frac{\det\left([x'-x]_N, u_{i_2}(y_2)[x'-y_2]_N,\dots,u_{i_k}(y_k)[x'-y_k]_N\right)}{|x'-x|^{n+s+1}|x'-y_2|^{n+s+1}\cdots|x'-y_k|^{n+s+1}}\;dy_2\cdots dy_k\,dx'\\
&= \int u_{i_k}(y_k)\cdots\int u_{i_2}(y_2)\int_{D^c_{\varepsilon_1,\dots,\varepsilon_k}}\\
&\qquad \frac{\det\left([x'-x]_N,[x'-y_2]_N,\dots,[x'-y_k]_N\right)}{|x'-x|^{n+s+1}|x'-y_2|^{n+s+1}\cdots|x'-y_k|^{n+s+1}}\;dx'\,dy_2\cdots dy_k.
\end{aligned}$$
Set
$$g(x,x',y_2,\dots,y_k) := \frac{\det\left([x'-x]_N,[x'-y_2]_N,\dots,[x'-y_k]_N\right)}{|x'-x|^{n+s+1}|x'-y_2|^{n+s+1}\cdots|x'-y_k|^{n+s+1}}.$$
Then,
$$\begin{aligned}
&\left|\int_{B(x,\varepsilon_1)^c} f^x_{\varepsilon_2,\dots,\varepsilon_k}(x')\,dx'\right| \leq\\
&\|u\|^{k-1}_\infty\int_{\operatorname{supp} u}\cdots\int_{\operatorname{supp} u}\left|\int_{D^c_{\varepsilon_1,\dots,\varepsilon_k}} g(x,x',y_2,\dots,y_k)\,dx'\right|dy_2\cdots dy_k.
\end{aligned} \tag{4.24}$$

Thanks to Lemma 4.1.1,

$$\left| \int_{D^c_{\varepsilon_1,\ldots,\varepsilon_k}} g(x, x', y_2, \ldots, y_k)\, dx' \right| \leq \frac{\epsilon_k^{1-s}}{(\epsilon_1 \cdots \epsilon_{k-1})^{n+s+2}} G(\epsilon_k, x - y_k, y_2 - y_k, \ldots, y_{k-1} - y_k), \tag{4.25}$$

where G is the function that appears therein. Integrating in (4.25), we find that

$$\int_{\operatorname{supp} u} \cdots \int_{\operatorname{supp} u} \left| \int_{D^c_{\varepsilon_1,\ldots,\varepsilon_k}} g(x, x', y_2, \ldots, y_k)\, dx' \right| dy_2 \cdots dy_k \leq h(\varepsilon_k, x)\, \frac{\epsilon_k^{1-s}}{(\epsilon_1 \cdots \epsilon_{k-1})^{n+s+2}},$$

for some continuous function $h : [0, \infty) \times \mathbb{R}^n \to [0, \infty)$. Consequently,

$$\lim_{\varepsilon_k \to 0} \int_{\operatorname{supp} u} \cdots \int_{\operatorname{supp} u} \left| \int_{D^c_{\varepsilon_1,\ldots,\varepsilon_k}} g(x, x', y_2, \ldots, y_k)\, dx' \right| dy_2 \cdots dy_k = 0,$$

and, in view of (4.23) and (4.24), we obtain that $\operatorname{div}^s[(\operatorname{cof}[D^s u]_M)_{i_1}]_{\bar{N}}(x) = 0$. □

4.2 Weak continuity of $\det D^s u$

In this section we prove that any minor (determinant of a submatrix) of $D^s u$ is a weakly continuous mapping in $H^{s,p}$. We start by expressing a nonlocal integration by parts formula for the minors of $D^s u$ that involves the operator K^s_φ of Lemma 3.3.2. Recall that for any $F \in \mathbb{R}^{n \times n}$ and $1 \leq i \leq n$ we denote by F_i the i-th row of F.

Lemma 4.2.1. *Let $k \in \mathbb{N}$ be with $1 \leq k \leq n$. Consider indices $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$ and the functions*

$$[\cdot]_M = [\cdot]_{M_{i_1,\ldots,i_k;j_1,\ldots,j_k}}, \quad [\cdot]_{\bar{M}} = [\cdot]_{\bar{M}_{i_1,\ldots,i_k;j_1,\ldots,j_k}}, \quad [\cdot]_{\tilde{N}} = [\cdot]_{\tilde{N}_{i_1,\ldots,i_k}}$$

of Definition 4.1.1. Let $p \geq k-1$, $q \geq \frac{p}{p-1}$ and $0 < s < 1$. Let $u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ be such that $\operatorname{cof}[D^s u]_M \in L^q(\mathbb{R}^n, \mathbb{R}^{k \times k})$. Then, $\det[D^s u]_M \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and for every $\varphi \in C^\infty_c(\mathbb{R}^n)$ we have that $[u]_{\tilde{N}} \cdot K^s_\varphi([\operatorname{cof}[D^s u]_M]_{\bar{M}}) \in L^1(\mathbb{R}^n)$ and

$$\int \det[D^s u]_M(x)\, \varphi(x)\, dx = -\frac{1}{k} \int [u]_{\tilde{N}}(x) \cdot K^s_\varphi([\operatorname{cof}[D^s u]_M]_{\bar{M}})(x)\, dx. \tag{4.26}$$

Proof. The fact $\det[D^s u]_M \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a consequence of formula (4.4) and Hölder's inequality, since $q \geq \frac{p}{p-1}$. Moreover, $[u]_{\tilde{N}} \cdot K^s_\varphi([\operatorname{cof}[D^s u]_M]_{\bar{M}}) \in L^1(\mathbb{R}^n)$, since $[u]_{\tilde{N}} \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $K^s_\varphi([\operatorname{cof}[D^s u]_M]_{\bar{M}}) \in L^r(\mathbb{R}^n, \mathbb{R}^n)$ for all $r \in [1, q]$ thanks to Lemma 3.3.2.

Assume first $u \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^n)$ and let $\psi \in C^\infty_c(\mathbb{R}^n)$. Fix $x \in \mathbb{R}^n$ and $i \in \{i_1, \ldots, i_k\}$. By Lemma 3.3.6 and Theorem 4.1.2,

$$\operatorname{div}^s\left(\psi\left([\operatorname{cof}[D^s u]_M]_{\bar{M}}\right)_i\right)(x) = K^s_\psi\left(\left([\operatorname{cof}[D^s u]_M]_{\bar{M}}\right)^T_i\right)(x).$$

When we apply Proposition 3.1.3, we obtain from integration of the previous formula that

$$0 = \int \operatorname{div}^s\left(\psi\left([\operatorname{cof}[D^s u]_M]_{\bar{M}}\right)_i\right)(x)\, dx = \int K^s_\psi\left(\left([\operatorname{cof}[D^s u]_M]_{\bar{M}}\right)^T_i\right)(x)\, dx.$$

By Fubini's theorem and the definitions of K^s_ψ and fractional gradient,

$$\int K^s_\psi\left(\left([\operatorname{cof}[D^s u]_M]_{\bar{M}}\right)^T_i\right)(x)\, dx = \int D^s\psi(y) \cdot \left([\operatorname{cof}[D^s u]_M]_{\bar{M}}\right)_i(y)\, dy.$$

We thus have the equality

$$\int D^s\psi(y) \cdot \left([\operatorname{cof}[D^s u]_M]_{\bar{M}}\right)_i(y)\, dy = 0. \tag{4.27}$$

Now we assume that $u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ with $\operatorname{cof}[D^s u]_M \in L^q(\mathbb{R}^n, \mathbb{R}^{k\times k})$, and, again $\psi \in C^\infty_c(\mathbb{R}^n)$. Taking into account Proposition 3.2.2, let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $C^\infty_c(\mathbb{R}^n, \mathbb{R}^n)$ converging to u in $H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$. Then $[D^s u_j]_M \to [D^s u]_M$ in $L^p(\mathbb{R}^n, \mathbb{R}^{k\times k})$ and, hence, $\operatorname{cof}[D^s u_j]_M$ converges to $\operatorname{cof}[D^s u]_M$ in $L^{\frac{p}{k-1}}(\mathbb{R}^n, \mathbb{R}^{k\times k})$, so $[\operatorname{cof}[D^s u_j]_M]_{\bar{M}} \to [\operatorname{cof}[D^s u]_M]_{\bar{M}}$ in $L^{\frac{p}{k-1}}(\mathbb{R}^n, \mathbb{R}^{n\times n})$. Therefore, (4.27) holds as well, since $D^s\psi \in L^r(\mathbb{R}^n)$ for all $r \in [1, \infty]$ (see Lemma 3.3.1). Now let $\psi \in H^{s,p}(\mathbb{R}^n)$ be of compact support, and let $\{\psi_j\}_{j\in\mathbb{N}}$ be a sequence in $C^\infty_c(\mathbb{R}^n)$ converging to ψ in $H^{s,p}(\mathbb{R}^n)$ such that $\bigcup_{j\in\mathbb{N}} \operatorname{supp}\psi_j$ is bounded. Then, by Lemma 3.3.3 and Remark 3.3.1, $D^s\psi_j \to D^s\psi$ in $L^r(\mathbb{R}^n)$ for all $r \in [1, p]$. As $[\operatorname{cof}[D^s u]_M]_{\bar{M}} \in L^q(\mathbb{R}^n, \mathbb{R}^{n\times n})$, we have that (4.27) holds as well. To sum up, formula (4.27) is valid for any $u \in H^{s,p}(\mathbb{R}^n)$ with $\operatorname{cof}[D^s u]_M \in L^q(\mathbb{R}^n, \mathbb{R}^{k\times k})$ and any $\psi \in H^{s,p}(\mathbb{R}^n)$ of compact support.

We apply (4.27) to $\psi = \varphi u_i$, which is in $H^{s,p}(\mathbb{R}^n)$ thanks to Lemma 3.3.6, and has compact support since so does φ. By the formula for $D^s\psi$ given by Lemma 3.3.6, we obtain that

$$\begin{aligned} 0 = &\int \varphi(y)\, D^s u_i(y) \cdot \left([\operatorname{cof}[D^s u]_M]_{\bar{M}}\right)_i(y)\, dy \\ &+ \int K^s_\varphi(u_i I)(y) \cdot \left([\operatorname{cof}[D^s u]_M]_{\bar{M}}\right)_i(y)\, dy. \end{aligned} \tag{4.28}$$

Using formula (4.4), the fact $i \in \{i_1, \ldots, i_k\}$ and elementary properties of the functions of Definition 4.1.1, we find that for any $F \in \mathbb{R}^{n\times n}$,

$$F_i \cdot ([\operatorname{cof}[F]_M]_{\bar{M}})_i = \det[F]_M .$$

Using this and Fubini's theorem, from (4.28) we arrive at

$$\begin{aligned}0 = &\int \varphi(y) \det[D^s u]_M(y)\, dy + \\ &c_{n,s} \int u_i(x) \int \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+s}} \left([\operatorname{cof}[D^s u]_M]_{\bar{M}}\right)_i(y) \cdot \frac{x-y}{|x-y|}\, dy\, dx.\end{aligned}$$

We sum this equality for $i = i_1, \ldots, i_k$ and obtain that

$$\begin{aligned}0 = &k \int \varphi(y) \det[D^s u]_M(y)\, dy + \\ &c_{n,s} \int [u]_{\tilde{N}}(x) \cdot \int \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+s}} \left([\operatorname{cof}[D^s u]_M]_{\bar{M}}\right)(y) \frac{x-y}{|x-y|}\, dy\, dx,\end{aligned}$$

which is the required formula. □

Now we establish the closedness and continuity properties of the minors of $D^s u$ in the weak topology of $H^{s,p}$. Recalling Definition 4.1.1 *a)*, a minor of order k is a function $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ such that there exist $1 \le i_1 < \cdots < i_k \le n$ and $1 \le j_1 < \cdots < j_k \le n$ for which $\mu(F) = \det[F]_M$ for all $F \in \mathbb{R}^{n\times n}$. Recall the notation p_s^* of Theorem 3.5.8, and the affine space $H_g^{s,p}$ of (3.16).

Theorem 4.2.2. *Let $p \ge n-1$ and $0 < s < 1$. Let $g \in H^{s,p}(\mathbb{R}^n)$ and $u \in H_g^{s,p}(\Omega, \mathbb{R}^n)$. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $H_g^{s,p}(\Omega, \mathbb{R}^n)$ such that $u_j \rightharpoonup u$ in $H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$. Then*

a) If $k \in \mathbb{N}$ with $1 \le k \le n-2$ and μ is a minor of order k then $\mu(D^s u_j) \rightharpoonup \mu(D^s u)$ in $L^{\frac{p}{k}}(\mathbb{R}^n)$ as $j \to \infty$.

b) If $\operatorname{cof} D^s u_j \rightharpoonup \vartheta$ in $L^q(\mathbb{R}^n, \mathbb{R}^{n\times n})$ for some $q \in [1,\infty)$ and $\vartheta \in L^q(\mathbb{R}^n, \mathbb{R}^{n\times n})$ then $\vartheta = \operatorname{cof} D^s u$.

c) Assume $\det D^s u_j \rightharpoonup \theta$ in $L^\ell(\mathbb{R}^n)$ for some $\ell \in [1,\infty)$ and some $\theta \in L^\ell(\mathbb{R}^n)$. If $sp < n$ assume, in addition, that $\operatorname{cof} D^s u_j \rightharpoonup \operatorname{cof} D^s u$ in $L^q(\mathbb{R}^n, \mathbb{R}^{n\times n})$ for some $q \in (\frac{p_s^*}{p_s^*-1}, \infty)$. Then $\theta = \det D^s u$.

Proof. We will prove *a)* by induction on k. For $k = 1$ the result is trivial. Assume it holds for some $k \le n-3$ and let us prove it for $k+1$. Let μ be a minor of order $k+1$. In the notation of Definition 4.1.1 *a)*, $\mu(F) =$

$\det[F]_M$ for all $F \in \mathbb{R}^{n\times n}$, where $[\cdot]_M = [\cdot]_{M_{i_1,\dots,i_{k+1};j_1,\dots,j_{k+1}}}$ for some $1 \leq i_1 < \dots < i_{k+1} \leq n$ and $1 \leq j_1 < \dots < j_{k+1} \leq n$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. By induction assumption, $\operatorname{cof}[D^s u_j]_M \rightharpoonup \operatorname{cof}[D^s u]_M$ in $L^{\frac{p}{k}}(\mathbb{R}^n, \mathbb{R}^{(k+1)\times(k+1)})$ as $j \to \infty$, so $[\operatorname{cof}[D^s u_j]_M]_{\bar{M}} \rightharpoonup [\operatorname{cof}[D^s u]_M]_{\bar{M}}$ in $L^{\frac{p}{k}}(\mathbb{R}^n, \mathbb{R}^{n\times n})$. By Lemma 3.3.2, $K^s_\varphi([\operatorname{cof}[D^s u_j]_M]_{\bar{M}}) \rightharpoonup K^s_\varphi([\operatorname{cof}[D^s u]_M]_{\bar{M}})$ in $L^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $r \in [1, \frac{p}{k}]$. By Theorem 3.5.14, $[u_j]_{\tilde{N}} \to [u]_{\tilde{N}}$ in $L^p(\mathbb{R}^n)$, so

$$[u_j]_{\tilde{N}} \cdot K^s_\varphi([\operatorname{cof}[D^s u_j]_M]_{\bar{M}}) \rightharpoonup [u]_{\tilde{N}} \cdot K^s_\varphi([\operatorname{cof}[D^s u]_M]_{\bar{M}}) \quad \text{in } L^1(\mathbb{R}^n) \tag{4.29}$$

since $\frac{k}{p} + \frac{1}{p} \leq 1$. We apply Lemma 4.2.1 and, in particular, formula (4.26) to conclude that

$$\int \det[D^s u_j(x)]_M \, \varphi(x) \, dx \to \int \det[D^s u(x)]_M \, \varphi(x) \, dx. \tag{4.30}$$

This shows that $\det[D^s u_j]_M \rightharpoonup \det[D^s u]_M$ in the sense of distributions. As $\{\det[D^s u_j]_M\}_{j\in\mathbb{N}}$ is bounded in $L^{\frac{p}{k+1}}(\mathbb{R}^n)$ and $p > k+1$, we have that $\det[D^s u_j]_M \rightharpoonup \det[D^s u]_M$ in $L^{\frac{p}{k+1}}(\mathbb{R}^n)$.

The proof of *b)* follows the lines of *a)*. Let μ be a minor of order $n-1$. In the notation of Definition 4.1.1 *a)*, $\mu(F) = \det[F]_M$ for all $F \in \mathbb{R}^{n\times n}$, where $[\cdot]_M = [\cdot]_{M_{i_1,\dots,i_{n-1};j_1,\dots,j_{n-1}}}$ for some $1 \leq i_1 < \dots < i_{n-1} \leq n$ and $1 \leq j_1 < \dots < j_{n-1} \leq n$. Let $\varphi \in C_c^\infty(\Omega)$. By part *a)*, $\operatorname{cof}[D^s u_j]_M \rightharpoonup \operatorname{cof}[D^s u]_M$ in $L^{\frac{p}{n-2}}(\mathbb{R}^n, \mathbb{R}^{(n-1)\times(n-1)})$, so $[\operatorname{cof}[D^s u_j]_M]_{\bar{M}} \rightharpoonup [\operatorname{cof}[D^s u]_M]_{\bar{M}}$ in $L^{\frac{p}{n-2}}(\mathbb{R}^n, \mathbb{R}^{n\times n})$. By Lemma 3.3.2, $K^s_\varphi([\operatorname{cof}[D^s u_j]_M]_{\bar{M}}) \rightharpoonup K^s_\varphi([\operatorname{cof}[D^s u]_M]_{\bar{M}})$ in $L^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $r \in [1, \frac{p}{n-2}]$. By Theorem 3.5.14, $[u_j]_{\tilde{N}} \to [u]_{\tilde{N}}$ in $L^p(\mathbb{R}^n)$, so convergence (4.29) is also valid since $\frac{n-2}{p} + \frac{1}{p} \leq 1$. Thanks to (4.26), we conclude that convergence (4.30) holds. This shows that $\mu(D^s u_j) \rightharpoonup \mu(D^s u)$ in the sense of distributions. As this is true for every minor μ of order $n-1$, we obtain that $\operatorname{cof} D^s u_j \rightharpoonup \operatorname{cof} D^s u$ in the sense of distributions. Thanks to the assumption, $\vartheta = \operatorname{cof} D^s u$.

We finally show part *c)*. Let $\varphi \in C_c^\infty(\Omega)$. Assume first $sp < n$. By the assumption and Lemma 3.3.2, $K^s_\varphi(\operatorname{cof} D^s u_j) \rightharpoonup K^s_\varphi(\operatorname{cof} D^s u)$ in $L^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $r \in [1, q]$. By Theorem 3.5.14, $u_j \to u$ in $L^t(\mathbb{R}^n)$ for every $t \in [1, p^*_s)$, so

$$u_j \cdot K^s_\varphi(\operatorname{cof} D^s u_j) \rightharpoonup u_j \cdot K^s_\varphi(\operatorname{cof} D^s u) \quad \text{in } L^1(\mathbb{R}^n) \tag{4.31}$$

since $\frac{1}{q} + \frac{1}{p^*_s} < 1$.

Assume now $sp \geq n$. Then $\{\operatorname{cof} D^s u_j\}_{j\in\mathbb{N}}$ is bounded in $L^{\frac{p}{n-1}}(\mathbb{R}^n, \mathbb{R}^{n\times n})$ so, thanks to part *b)*, $\operatorname{cof} D^s u_j \rightharpoonup \operatorname{cof} D^s u$ in $L^{\frac{p}{n-1}}(\mathbb{R}^n, \mathbb{R}^{n\times n})$. By Lemma 3.3.2, $K^s_\varphi(\operatorname{cof} D^s u_j) \rightharpoonup K^s_\varphi(\operatorname{cof} D^s u)$ in $L^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $r \in [1, \frac{p}{n-1}]$. By

Theorem 3.5.14, $u_j \to u$ in $L^t(\mathbb{R}^n)$ for every $t \in [1, \infty)$, so convergence (4.31) holds since $p > n - 1$.

In either case, we have convergence (4.31), so by (4.26) we obtain

$$\int \det D^s u_j(x)\, \varphi(x)\, dx \to \int \det D^s u(x)\, \varphi(x)\, dx.$$

This shows that $\det D^s u_j \rightharpoonup \det D^s u$ in the sense of distributions, so $\theta = \det D^s u$. □

Remark 4.2.1. *A natural question is whether the weak continuity of the determinant of the fractional gradient may be concluded as a consequence of the weak continuity of the determinant of the classical gradient. Indeed, one can use the properties of the Riesz potential to give a simpler proof in the case $p > n$. To be precise, in [38, Prop. 2.2] (see also [99, Th. 1.2]) it is shown that*

$$D^s u = D(I_{1-s} * u) \tag{4.32}$$

for any $u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$, where $I_{1-s}(x) = \frac{-c_{n,s}}{n+s-1}|x|^{-(n+1-s)}$. Now, writing the determinant as a divergence [10, Sect. 6] and using (4.32), we have that

$$\int \det D^s u\, \varphi\, dx = -\int (I_{1-s} * u) \cdot (\operatorname{cof} D^s u\, D\varphi)\, dx, \tag{4.33}$$

for any $u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and any test function $\varphi \in C_c^\infty(\mathbb{R}^n)$. By density of C_c^∞ in $H^{s,p}$, equality (4.33) holds for any $u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$. Now, taking into account the Hardy–Littlewood–Sobolev embedding [110, Th. 1, b)], and Theorem 3.5.8 it is easy to obtain the weak continuity of $\det D^s u$ in $H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ for $p > n$. We do not know whether it is possible to extend the previous argument for $p \geq n-1$ without making use of the fractional Piola identity. This is relevant since, at least in the classical case, the condition $p \geq n - 1$ is sought in order to have more realistic assumptions (in Solid Mechanics) than $p > n$.

4.3 Existence of minimizers and equilibrium equations

In this section we prove the existence of minimizers in $H^{s,p}$ of functionals of the form

$$I(u) := \int W(x, u(x), D^s u(x))\, dx. \tag{4.34}$$

under natural coercivity and polyconvexity assumptions. We also derive the associated Euler–Lagrange equation, which is a fractional partial differential system of equations.

We will assume the hypothesis of polyconvexity, introduced in Definition 4.0.1.

The existence theorem is the following. Its proof relies on a standard argument in the Calculus of Variations, once we have the continuity (with respect to the weak convergence) of the minors given by Theorem 4.2.2.

Theorem 4.3.1. *Let $p \geq n-1$ satisfy $p > 1$ and $0 < s < 1$. Let $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ satisfy the following conditions:*

a) W is $\mathcal{L}^n \times \mathcal{B}^n \times \mathcal{B}^{n\times n}$-measurable, where $\mathcal{L}^n$ denotes the Lebesgue sigma-algebra in $\mathbb{R}^n$, whereas $\mathcal{B}^n$ and $\mathcal{B}^{n\times n}$ denote the Borel sigma-algebras in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, respectively.

b) $W(x,\cdot,\cdot)$ is lower semicontinuous for a.e. $x \in \mathbb{R}^n$.

c) For a.e. $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}^n$, the function $W(x,y,\cdot)$ is polyconvex.

d) There exist a constant $c > 0$, an $a \in L^1(\mathbb{R}^n)$ and a Borel function $h : [0,\infty) \to [0,\infty)$ such that

$$\lim_{t\to\infty} \frac{h(t)}{t} = \infty$$

and, for some $q > \frac{p_s^*}{p_s^*-1}$, if $sp < n$,

$$\begin{cases} W(x,y,F) \geq a(x) + c\,|F|^p + c\,|\operatorname{cof} F|^q + h(|\det F|), & \text{if } sp < n, \\ W(x,y,F) \geq a(x) + c\,|F|^p\,, & \text{if } sp \geq n, \end{cases}$$

for a.e. $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$ and all $F \in \mathbb{R}^{n\times n}$.

Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $u_0 \in H^{s,p}(\mathbb{R}^n,\mathbb{R}^n)$. Define I as in (4.34), and assume that I is not identically infinity in $H^{s,p}_{u_0}(\Omega,\mathbb{R}^n)$. Then there exists a minimizer of I in $H^{s,p}_{u_0}(\Omega,\mathbb{R}^n)$.

Proof. Assumption *d)* shows that the functional I is bounded below by $\int a$. As I is not identically infinity in $H^{s,p}_{u_0}(\Omega,\mathbb{R}^n)$, there exists a minimizing sequence $\{u_j\}_{j\in\mathbb{N}}$ of I in $H^{s,p}_{u_0}(\Omega,\mathbb{R}^n)$. Assumption *d)* implies that $\{D^s u_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\mathbb{R}^n,\mathbb{R}^{n\times n})$. Thanks to Theorem 3.5.8, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega,\mathbb{R}^{n\times n})$. As $u_j = u_0$ in Ω^c for all $j \in \mathbb{N}$, we also have that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\mathbb{R}^n,\mathbb{R}^n)$, and, consequently, also in $H^{s,p}(\mathbb{R}^n,\mathbb{R}^n)$. Then, we can extract a weakly convergent subsequence since $H^{s,p}(\mathbb{R}^n,\mathbb{R}^n)$ is reflexive. Using Theorem 3.5.14, we obtain that there exists $u \in H^{s,p}_{u_0}(\mathbb{R}^n,\mathbb{R}^n)$ such that for a subsequence (not relabelled),

$$u_j \rightharpoonup u \text{ in } H^{s,p}(\mathbb{R}^n,\mathbb{R}^n) \quad \text{and} \quad u_j \to u \text{ in } L^p(\mathbb{R}^n,\mathbb{R}^n). \tag{4.35}$$

Now, by Theorem 4.2.2, for any minor μ of order $k \leq n-2$, we have that

$$\mu(D^s u_j) \rightharpoonup \mu(D^s u) \text{ in } L^{\frac{p}{k}}(\mathbb{R}^n). \tag{4.36}$$

If $sp < n$ then, $\{\operatorname{cof} D^s u_j\}_{j\in\mathbb{N}}$ is bounded in $L^q(\mathbb{R}^n, \mathbb{R}^{n\times n})$ by assumption *d)*, whereas if $sp \geq n$ we call $q := \frac{p}{n-1}$ and have that $\{\operatorname{cof} D^s u_j\}_{j\in\mathbb{N}}$ is bounded in $L^q(\mathbb{R}^n, \mathbb{R}^{n\times n})$. In either case we have that $q > 1$, so for a subsequence $\{\operatorname{cof} D^s u_j\}_{j\in\mathbb{N}}$ converges weakly in $L^q(\mathbb{R}^n, \mathbb{R}^{n\times n})$ and, by Theorem 4.2.2,

$$\operatorname{cof} D^s u_j \rightharpoonup \operatorname{cof} D^s u \text{ in } L^q(\mathbb{R}^n, \mathbb{R}^{n\times n}). \tag{4.37}$$

If $sp < n$ then, by assumption *d)* and de la Vallée Poussin's criterion, $\{\det D^s u_j\}_{j\in\mathbb{N}}$ is equiintegrable, whereas if $sp \geq n$ we have that $\{\det D^s u_j\}_{j\in\mathbb{N}}$ is bounded in $L^{\frac{p}{n}}(\mathbb{R}^n)$ and $\frac{p}{n} > 1$. In either case we have that, for a subsequence $\{\det D^s u_j\}_{j\in\mathbb{N}}$ converges weakly in $L^{\ell}(\mathbb{R}^n)$ with

$$\begin{cases} \ell = 1 & \text{if } sp < n, \\ \ell = \frac{p}{n} & \text{if } sp \geq n, \end{cases}$$

and, hence, by Theorem 4.2.2,

$$\det D^s u_j \rightharpoonup \det D^s u \text{ in } L^{\ell}(\mathbb{R}^n). \tag{4.38}$$

Convergences (4.35)–(4.38) imply, thanks to a standard lower semicontinuity result for polyconvex functionals (see, e.g., [15, Th. 5.4] or [59, Th. 7.5]), that for any $R > 0$,

$$\int_{B(0,R)} W(x, u(x), D^s u(x))\, dx \leq \liminf_{j\to\infty} \int_{B(0,R)} W(x, u_j(x), D^s u_j(x))\, dx.$$

Therefore,

$$\begin{aligned} &\int_{B(0,R)} (W(x, u(x), D^s u(x)) - a(x))\, dx \leq \\ &\liminf_{j\to\infty} \int_{B(0,R)} (W(x, u_j(x), D^s u_j(x)) - a(x))\, dx \leq \\ &\liminf_{j\to\infty} \int (W(x, u_j(x), D^s u_j(x)) - a(x))\, dx. \end{aligned}$$

By monotone convergence,

$$\int (W(x, u(x), D^s u(x)) - a(x))\, dx \leq \liminf_{j\to\infty} \int (W(x, u_j(x), D^s u_j(x)) - a(x))\, dx,$$

so

$$I(u) \leq \liminf_{j\to\infty} I(u_j).$$

Therefore, u is a minimizer of I in $H^{s,p}_{u_0}(\Omega, \mathbb{R}^n)$ and the proof is concluded. □

Comparing Lemmas 3.6.1 and 3.6.2 with Theorem 4.3.1, we see that functions exhibiting singularities as those shown in those lemmas are compatible with the existence result of Theorem 4.3.1, in opposition to the case of classical elasticity (see, e.g., [10, 11, 13, 14, 16, 69]). Indeed, for a $u \in H^{s,p}(\mathbb{R}^n)$ of compact support and $p > n$, by Hölder's inequality and Remark 3.3.1, $\operatorname{cof} D^s u \in L^q(\mathbb{R}^n, \mathbb{R}^{n\times n})$ for every $q \in [1, \frac{p}{n-1}]$ and $\det D^s u \in L^r(\mathbb{R}^n)$ for every $r \in [1, \frac{p}{n}]$. Take now an $s \in (0,1)$ such that $sp < n$, so that this regime is compatible with cavitation (see Lemma 3.6.2). Considering the function h of Theorem 4.3.1 as $h(t) := t^{\frac{p}{n}}$, we see that this map u is compatible with the assumptions of Theorem 4.3.1 if and only if $\frac{p}{n-1} > \frac{p_s^*}{p_s^*-1}$, so $n^2 - np < sp$. To sum up, in the regime

$$p > n, \qquad 0 < s < \frac{n}{p}$$

a typical cavitation map is compatible with the hypothesis of Theorem 4.3.1. Similarly, if $p > n$ and $n^2 - np < sp < 1$, i.e., in the regime

$$p > n, \qquad 0 < s < \frac{1}{p}$$

the hypothesis of Theorem 4.3.1 are compatible with discontinuities along hypersurfaces.

To finish this section, we explore the equilibrium conditions that minimizers of functional (4.34) satisfy. This Euler–Lagrange, or equilibrium, conditions constitute a nonlinear system of fractional PDE, and therefore we are providing an existence result for such kind of systems based on polyconvexity. To be precise, given $g \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$, the boundary value problem reads as

$$\begin{cases} \operatorname{div}^s \left(\frac{\partial W}{\partial F}(x, u, D^s u)\right) = \frac{\partial W}{\partial u}(x, u, D^s u), & \text{in } \Omega \\ u = g & \text{in } \Omega^c. \end{cases} \tag{4.39}$$

As a consequence of the fractional integration by parts(Proposition 3.2.4), we define a weak solution of (4.39) as a $u \in H_g^{s,p}(\Omega, \mathbb{R}^n)$ satisfying

$$\int \left[\frac{\partial W}{\partial F}(x, u, D^s u) \cdot D^s v + \frac{\partial W}{\partial u}(x, u, D^s u) \cdot v\right] dx = 0 \tag{4.40}$$

for all $v \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $v = 0$ in Ω^c.

The derivation of (4.40) for a minimizer u is standard. For this, we make the assumptions a–b) below, which are slightly adapted from [39, Conditions 3.22 and 3.33], although other sets of assumptions are also possible (see, e.g., [10, Sect. 7] or [39, Sect. 3.4.2]).

Theorem 4.3.2. *Let $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ be a function satisfying*

a) $W(\cdot, u, F)$ is measurable for every $(u, F) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$ and $W(x, \cdot, \cdot)$ is of class C^1 for a.e. $x \in \mathbb{R}^n$.

b) There exist an $a \in L^1(\mathbb{R}^n)$, an $\alpha \in \mathbb{R}$ with

$$\begin{cases} \alpha \in [p, p_s^*] & \text{if } sp < n, \\ \alpha \in [p, \infty) & \text{if } sp \geq n, \end{cases}$$

and a $c > 0$ such that

$$|W(x, u, F)| + \left| \frac{\partial W}{\partial u}(x, u, F) \right| + \left| \frac{\partial W}{\partial F}(x, u, F) \right| \leq a(x) + c \left(|u|^\alpha + |F|^p \right),$$

for a.e. $x \in \mathbb{R}^n$ and all $(u, F) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$.

Let $g \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$. Define I as in (4.34), and let u be a minimizer of I in $H^{s,p}_g(\Omega, \mathbb{R}^n)$. Then u is a weak solution of (4.39).

Proof. Let us fix $v \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $v = 0$ in Ω^c. As $u + \tau v \in H^{s,p}_g(\Omega, \mathbb{R}^n)$ for any $\tau \in \mathbb{R}$, it suffices to show that the derivative of $I(u + \tau\, v)$ exists at $\tau = 0$ and equals the left hand side of (4.40). Thanks to the dominated convergence theorem, it suffices to show (see, e.g., [74, Ch. 13, §2, Lemma 2.2]) that there exists $G \in L^1(\mathbb{R}^n)$ such that for every $\tau \in \mathbb{R}$ with $|\tau| < 1$ we have

$$I(u + \tau\, v) < \infty \tag{4.41}$$

and

$$\left| \frac{d}{d\tau} W(x, u(x) + \tau v(x), D^s u(x) + \tau D^s v(x)) \right| \leq G(x), \qquad \text{a.e. } x \in \mathbb{R}^n. \tag{4.42}$$

Let us check condition (4.41). Thanks to b) ,

$$I(u + \tau\, v) \leq \int a + C \int \left(|u|^\alpha + |D^s u|^p + |v|^\alpha + |D^s v|^p \right)$$

for some constant $C > 0$. Clearly, the integral of $|v|^\alpha$ is finite since $v \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$, and so is the integral of $|D^s v|^p$ due to Lemma 3.3.1. In addition, the integral of $|D^s u|^p$ is finite because $u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$. Now, by Theorem 3.5.8 and the interpolation (or Hölder) inequality, $u \in L^r(\mathbb{R}^n, \mathbb{R}^n)$ for all $r \in [p, p_s^*]$ if $sp < n$, and for all $r \in [p, \infty)$ if $sp \geq n$. Therefore, $|u|^\alpha \in L^1(\mathbb{R}^n)$. Condition (4.41) is thus satisfied.

We now show condition (4.42). We have, for $|\tau| < 1$ and a.e. $x \in \Omega$,

$$\begin{aligned}&\left|\frac{d}{d\tau}W(x, u(x) + \tau v(x), D^s u(x) + \tau D^s v(x))\right| \leq \\ &\left|\frac{\partial W}{\partial u}(x, u(x) + \tau v(x), D^s u(x) + \tau D^s v(x))\right| \|v\|_{L^\infty} + \\ &\left|\frac{\partial W}{\partial F}(x, u(x) + \tau v(x), D^s u(x) + \tau D^s v(x))\right| \|D^s v\|_{L^\infty} ,\end{aligned}$$

where we have used Lemma 3.3.1 to show that $D^s v \in L^\infty(\mathbb{R}^n)$. Now, by b),

$$\begin{aligned}&\left|\frac{\partial W}{\partial u}(x, u(x) + \tau v(x), D^s u(x) + \tau D^s v(x))\right| + \\ &\left|\frac{\partial W}{\partial F}(x, u(x) + \tau v(x), D^s u(x) + \tau D^s v(x))\right| \leq \\ &\qquad a(x) + C\left(|u(x)|^\alpha + |v(x)|^\alpha + |D^s u(x)|^p + |D^s v(x)|^p\right),\end{aligned} \tag{4.43}$$

for some constant $C > 0$. As before, the right hand side of (4.43) is in $L^1(\mathbb{R}^n)$, so condition (4.42) is proved. $\square$

Chapter 5.

Γ- convergence of polyconvex functionals depending on the fractional gradient when s goes to 1

In this section we continue with the study of polyconvex functionals depending on the s-fractional gradient by further exploring it through the study of its limit when $s \nearrow 1$. The main results of this part (shown in [22]) are described as follows. We prove the strong convergence in L^p of $D^s u$ to Du for functions $u \in W^{1,p}$, generalizing, and making the topology precise, the convergence mentioned at the introduction of Chapter 3 for smooth functions. Notice that this convergence is performed in the fractional parameter s rather than in the *horizon*, as done in [84]. This result is of interest in itself, as it provides a precise differential object converging to the distributional gradient. We also show a weak compactness result in $W^{1,p}$, establishing that if $\{u_s\}$ is a sequence such that $\{D^s u_s\}$ is bounded in L^p, then there exists a $u \in W^{1,p}$ such that u_s converges strongly and $D^s u_s$ converges weakly in L^p as $s \nearrow 1$ to u and Du, respectively. We also show the weak convergence of the minors of $D^s u_s$ to those of Du, whenever $D^s u_s$ converges weakly in L^p to Du; as a consequence, we establish a new semicontinuity result for polyconvex functionals. Finally, we show that the family of vector variational problems based on minimization of

$$\mathcal{I}_s(u) = \int_{\mathbb{R}^n} W(x, u(x), D^s u(x))\, dx, \qquad u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$$

Γ-converges (see Subsection 1.2.3) to the functional

$$\mathcal{I}(u) = \int_{\mathbb{R}^n} W(x, u(x), Du(x))\, dx, \qquad u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$$

as $s \nearrow 1$, under the essential assumption of polyconvexity of $W(x, u, \cdot)$ (Definition 4.0.1) ; we also need the extra assumption $p > n$ for the Γ-convergence. Other references dealing with Γ-convergence of variational functionals in the nonlocal setting are [91] (in the context of $W^{s,p}$), [25] (in nonlinear peridynamics) and [82] (in linear and geometrically nonlinear peridynamics). Concurrently and independently to the work developed in this Chapter, [37] treated a closely related problem. Essentially, they address somewhat similar Γ-convergence questions than us but mainly in the special case $p = 1$ and without dealing with polyconvexity. More precisely, they prove some Γ-convergence properties in the space $BV^s(\mathbb{R}^n)$ of functions with bounded s-fractional variation, and study the fractional operators involved. Interestingly, they provide a different proof, not based on Fourier transform like our Theorem 5.1.1, of the convergence of $D^s u$ to Du in L^p for $u \in W^{1,p}$.

The outline of this Chapter is as follows. In Section 5.1 we prove the localization of the fractional gradient; namely, that $D^s u$ converges to Du in L^p for a fixed $u \in W^{1,p}$ (Theorem 5.1.1). In Section 5.2 we prove the compactness result: for any sequence $\{u_s\}$ with a fixed complementary-value data and bounded $\{D^s u_s\}$ in L^p, there exist $u \in W^{1,p}$ and a subsequence strongly convergent to u in L^p, with the s-fractional gradients converging weakly to Du in L^p (Theorem 5.2.2). Section 5.3 is devoted to the weak convergence of the minors of $D^s u_s$ to the minors of Du when $D^s u_s$ converges weakly to Du in L^p (Theorem 5.3.2). Finally, in Section 5.4 we prove a novel semicontinuity result for polyconvex functionals (Theorem 5.4.1) and the Γ-convergence result of $\mathcal{I}_s$ to $\mathcal{I}$ under the assumption of polyconvexity (Theorem 5.4.2) and convexity (Theorem 5.4.3).

5.1 Localization of fractional gradients

In this section we prove the convergence of the s-fractional gradient of a $W^{1,p}$ function to its local gradient as $s \nearrow 1$. This result is to be expected, and easy to obtain for smooth functions using the Fourier transform (see Lemma 3.1.7). In this section we provide a complete proof for functions in $W^{1,p}(\mathbb{R}^n)$. This result, which is of interest in its own right, is a first step to prove the Γ-convergence of the functional $\mathcal{I}_s$ to $\mathcal{I}$ (see the Introduction). It should be compared with [28, Cor. 2], where the convergence of the Gagliardo seminorm to the L^p norm of the fractional gradient is shown (see also [92, Prop. 15.7]).

The main result of the section is the following. As mentioned previously, a similar result has been simultaneously proved in [37, Sect. 4.1] without the use of Fourier transform.

Theorem 5.1.1. *Let $0 < s < 1$ and $1 < p < \infty$. Then, for each $u \in W^{1,p}(\mathbb{R}^n)$,*

$$D^s u \to Du \text{ in } L^p(\mathbb{R}^n) \text{ as } s \nearrow 1.$$

Proof. We first prove the result for smooth functions and then extend it by density to $W^{1,p}(\mathbb{R}^n)$.

Let $u \in C_c^\infty(\mathbb{R}^n)$. By Lemma 3.1.7,

$$\widehat{D^s u}(\xi) = \frac{2\pi i \xi}{|2\pi\xi|^{1-s}} \hat{u}(\xi), \qquad \xi \in \mathbb{R}^n,$$

so by the elementary inequality $t^s \leq 1 + t$ for all $t \geq 0$,

$$\left| \widehat{D^s u}(\xi) \right| = |2\pi\xi|^s \, |\hat{u}(\xi)| \leq (1 + |2\pi\xi|) \, |\hat{u}(\xi)| \, . \tag{5.1}$$

As $\hat{u}$ is in the Schwartz space (because $u \in C_c^\infty(\mathbb{R}^n)$), both $\hat{u}$ and $\xi \, \hat{u}(\xi)$ are in $L^1(\mathbb{R}^n)$. Therefore, $\widehat{D^s u} \in L^1(\mathbb{R}^n)$. On the other hand, by basic properties of the Fourier transform, $\widehat{Du}(\xi) = 2\pi i \xi \hat{u}(\xi)$, so clearly, $\widehat{D^s u} \to \widehat{Du}$ a.e. as $s \nearrow 1$. Thanks to the bound (5.1) and dominated convergence, $\widehat{D^s u} \to \widehat{Du}$ in $L^1(\mathbb{R}^n)$. As the inverse Fourier transform is continuous from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, we also have that

$$D^s u \to Du \;\text{ uniformly in } \mathbb{R}^n.$$

Now, using a standard interpolation inequality (or Hölder's), we get that

$$\begin{aligned}
\|D^s u - Du\|_p &\leq \|D^s u - Du\|_1^{\frac{1}{p}} \|D^s u - Du\|_\infty^{\frac{1}{p'}} \\
&\leq (\|D^s u\|_1 + \|Du\|_1)^{\frac{1}{p}} \, \|D^s u - Du\|_\infty^{\frac{1}{p'}} \\
&\leq C \|u\|_{W^{1,1}(\mathbb{R}^n)}^{\frac{1}{p}} \|D^s u - Du\|_\infty^{\frac{1}{p'}},
\end{aligned}$$

where we have used Proposition 3.5.11, considering that, as $s \nearrow 1$, we can assume $s \geq \frac{1}{2}$, so the constant $C > 0$ does not depend on s. Thus, the convergence $D^s u \to Du$ in L^p follows and the result is true for C_c^∞ functions.

To conclude the proof, we extend this result through a density argument. Let us consider $u \in W^{1,p}(\mathbb{R}^n)$. Then, for every $\varepsilon > 0$ we can find $v \in C_c^\infty(\mathbb{R}^n)$ such that $\|v - u\|_{W^{1,p}(\mathbb{R}^n)} < \varepsilon$. Thus,

$$\begin{aligned}
\|D^s u - Du\|_p &\leq \|D^s u - D^s v\|_p + \|D^s v - Dv\|_p + \|Dv - Du\|_p \\
&\leq (C+1)\varepsilon + \|D^s v - Dv\|_p,
\end{aligned}$$

where we have used again Proposition 3.5.11. Finally, when we take limits we obtain that

$$\limsup_{s\nearrow 1} \|D^s u - Du\|_p \leq (C+1)\varepsilon,$$

for every $\varepsilon > 0$, which concludes the result. $\square$

Thanks to Lemma 3.2.3, the previous result also implies the convergence in L^p of the fractional divergence.

Corollary 5.1.2. *Let $0 < s < 1$ and $1 < p < \infty$. Then, for each $\phi \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$,*

$$\operatorname{div}^s \phi \to \operatorname{div} \phi \text{ in } L^p(\mathbb{R}^n) \text{ as } s \nearrow 1.$$

By duality we also have next corollary. Hence, interestingly, last convergence can be extend in a broader sense, a least in the sense of distribution, showing a certain convergence of more general sequences, including those with functions that may have singularities.

Corollary 5.1.3. *Let $0 < s < 1$ and $1 \leq p \leq \infty$. Then, for each $u \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, $\mathcal{D}^s u$ converges to $\mathcal{D}u$ in the sense of distributions D', where $\mathcal{D}u$ denotes the distributional derivative. I.e.*

$$\int \mathcal{D}^s u(x)\varphi(x)\,dx \to \int \mathcal{D}u(x)\varphi(x)\,dx$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$.

5.2 Compactness

In this section we establish that any sequence $\{u_s\}_{s\in(0,1)}$ with bounded $H_g^{s,p}(\Omega)$ norm is precompact in $L^q(\mathbb{R}^n)$ for a suitable $q \geq 1$.

Even though the continuous embedding of $H^{s,p}$ into $H^{\bar{s},p}$ for $0 < \bar{s} < s < 1$ is already known, we start by giving a new proof of this result, where we show that the embedding constant is independent of s. This proof follows the ideas of Theorem 3.5.9.

Proposition 5.2.1. *Let $0 < \bar{s} < s_0 < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then, there exists a constant $C = C(\Omega, n, s_0, \bar{s}) > 0$ such that for every $s \in [s_0, 1)$, $1 < p < \infty$ and $u \in H_0^{s,p}(\Omega)$ we have*

$$\|D^{\bar{s}} u\|_p \leq C \|D^s u\|_p. \tag{5.2}$$

Proof. By density, it is enough to prove the inequality for $u \in C_c^\infty(\Omega)$. We divide the proof into two steps.

Step 1. First, we prove that there exists $C = C(\Omega, n, s_0, \bar{s}) > 0$ such that

$$\|D^{\bar{s}} u\|_{L^p(\Omega)} \leq C \|D^s u\|_p. \tag{5.3}$$

Let $R \geq 1$ be such that $\Omega \subset B(0, R)$. Define $\Omega_1 := B(0, 2R)$ and fix $x \in \Omega$. Notice that, as a consequence of [99, Th. 1.2] and the semigroup property of the Riesz potential, we can write

$$D^{\bar{s}} u = I_{1-\bar{s}} * Du = (I_{1-s} * I_{s-\bar{s}}) * Du = I_{s-\bar{s}} * D^s u.$$

This equality, together with (3.15) and Lemma 3.1.9 yields

$$\begin{aligned} \left|D^{\bar{s}} u(x)\right| &\leq \frac{1}{\gamma(s-\bar{s})} \left[\int_{\Omega_1} \frac{|D^s u(y)|}{|x-y|^{n-(s-\bar{s})}}\, dy + \int_{\Omega_1^c} \frac{|D^s u(y)|}{|x-y|^{n-(s-\bar{s})}}\, dy\right] \\ &\leq C(n) \left[\int_{\Omega_1} \frac{|D^s u(y)|}{|x-y|^{n-(s-\bar{s})}}\, dy + \int_{\Omega_1^c} \frac{|D^s u(y)|}{|x-y|^{n-(s-\bar{s})}}\, dy\right]. \end{aligned} \tag{5.4}$$

Now $\Omega_1 \subset B(x, 3R)$, so

$$\begin{aligned} \int_{\Omega_1} \frac{1}{|x-y|^{n-(s-\bar{s})}}\, dy &\leq \int_{B(x,3R)} \frac{1}{|x-y|^{n-(s-\bar{s})}}\, dy = \frac{\sigma_{n-1}}{s-\bar{s}} (3R)^{(s-\bar{s})} \\ &\leq C(n) \frac{1}{s-\bar{s}} R. \end{aligned} \tag{5.5}$$

Similarly, $\Omega \subset B(y, 3R)$ for every $y \in \Omega_1$, so

$$\int_{\Omega} \frac{1}{|x-y|^{n-(s-\bar{s})}}\, dx \leq C(n) \frac{1}{s-\bar{s}} R. \tag{5.6}$$

By (5.5) and Hölder's inequality,

$$\int_{\Omega_1} \frac{|D^s u(y)|}{|x-y|^{n-(s-\bar{s})}}\, dy \leq \left[C(n) \frac{1}{s-\bar{s}} R\right]^{\frac{1}{p'}} \left(\int_{\Omega_1} \frac{|D^s u(y)|^p}{|x-y|^{n-(s-\bar{s})}}\, dy\right)^{\frac{1}{p}}.$$

Therefore, using (5.6) and Fubini's theorem, we find, as in (3.47),

$$\left[\int_{\Omega} \left(\int_{\Omega_1} \frac{|D^s u(y)|}{|x-y|^{n-(s-\bar{s})}}\, dy\right)^p dx\right]^{\frac{1}{p}} \leq C(n) \frac{1}{s-\bar{s}} R \left\|D^s u\right\|_p. \tag{5.7}$$

Now, for any $y \in \Omega_1^c$, similarly to (3.50), for each $x \in \Omega$ we have

$$\frac{1}{|x-y|^{n-(s-\bar{s})}} \leq C(n) \frac{1}{|y|^{n-(s-\bar{s})}} \tag{5.8}$$

and, in fact, (3.51) also holds. Thus, using (5.8) and (3.51),

$$\int_{\Omega_1^c} \frac{|D^s u(y)|}{|x-y|^{n-(s-\bar{s})}}\, dy \leq C(n)|\Omega|^{\frac{1}{p'}} \|u\|_{L^p(\Omega)} \int_{\Omega_1^c} \frac{1}{|y|^{n+s}} \frac{1}{|y|^{n-(s-\bar{s})}}\, dy$$
$$= C(n)|\Omega|^{\frac{1}{p'}} \frac{R^{-n-\bar{s}}}{n+\bar{s}} \|u\|_{L^p(\Omega)}\, .$$

This last inequality, combined with (5.4) and (5.7), implies by the triangular inequality that

$$\left\|D^{\bar{s}}u\right\|_{L^p(\Omega)} \leq C(n)\frac{1}{s-\bar{s}}R\|D^s u\|_p + |\Omega|C(n)\frac{R^{-n-\bar{s}}}{n+\bar{s}}\|u\|_{L^p(\Omega)}.$$

Finally, we apply Theorem 3.5.9 on the right hand side to obtain

$$\left\|D^{\bar{s}}u\right\|_{L^p(\Omega)} \leq C(n,\Omega)\left(\frac{1}{s-\bar{s}} + \frac{R^{-n-\bar{s}}}{n+\bar{s}}\frac{1}{s}\right)\|D^s u\|_p$$
$$\leq C(n,\Omega)\left(\frac{1}{s_0-\bar{s}} + \frac{1}{s_0(n+\bar{s})}\right)\|D^s u\|_p,$$

which completes the proof of (5.3).

Step 2. Now we prove (5.2). Let us call $\Omega_C = \Omega + B(0,1)$. Then,

$$\left\|D^{\bar{s}}u\right\|_p \leq \left\|D^{\bar{s}}u\right\|_{L^p(\Omega_C)} + \left\|D^{\bar{s}}u\right\|_{L^p(\Omega_C^c)}\, .$$

By (5.3) there exists $C > 0$ (depending on $\Omega_C, n, s_0, \bar{s}$, so, ultimately, on $\Omega, n, s_0, \bar{s}$) such that

$$\|D^{\bar{s}}u\|_p \leq C\|D^s u\|_p + \|D^{\bar{s}}u\|_{L^p(\Omega_C^c)}. \tag{5.9}$$

Now, for $x \in \Omega_C^c$,

$$D^{\bar{s}}u(x) = -c_{n,\bar{s}} \int_\Omega \frac{u(y)}{|x-y|^{n+\bar{s}}} \frac{x-y}{|x-y|} dy,$$

so, by Lemma 3.1.9,

$$\left|D^{\bar{s}}u(x)\right| \leq C(n) \int_\Omega \frac{|u(y)|}{|x-y|^{n+\bar{s}}} dy,$$

and, hence, by Minkowski's integral inequality,

$$\begin{aligned}\left\|D^{\bar{s}}u\right\|_{L^p(\Omega_C^c)} &\leq C(n)\left(\int_{\Omega_C^c}\left(\int_\Omega \frac{|u(y)|}{|x-y|^{n+\bar{s}}}dy\right)^p dx\right)^{\frac{1}{p}} \\ &\leq C(n)\int_\Omega |u(y)|\left(\int_{\Omega_C^c} \frac{1}{|x-y|^{(n+\bar{s})p}}dx\right)^{\frac{1}{p}} dy.\end{aligned} \tag{5.10}$$

Now, for every $y \in \Omega$ we have $\Omega_C^c - y \subset B(0,1)^c$, and hence

$$\int_{\Omega_C^c} \frac{1}{|x-y|^{(n+\bar{s})p}}\,dx = \int_{\Omega_C^c - y} \frac{1}{|z|^{(n+\bar{s})p}}\,dx \leq \int_{B(0,1)^c} \frac{1}{|z|^{(n+\bar{s})p}}\,dx$$
$$= \frac{\sigma_{n-1}}{(n+\bar{s})p-n} \leq \frac{\sigma_{n-1}}{\bar{s}}.$$

Thus, continuing from (5.10) we find that

$$\begin{aligned}\left\|D^{\bar{s}}u\right\|_{L^p(\Omega_C^c)} &\leq C(n)\max\{1, \frac{\sigma_{n-1}}{\bar{s}}\}\int_\Omega |u(y)|dy \\ &\leq C(n)\max\{1, \frac{\sigma_{n-1}}{\bar{s}}\}\max\{1,|\Omega|\}\,\|u\|_{L^p(\Omega)}\,.\end{aligned} \tag{5.11}$$

Inequalities (5.9), (5.11) and Theorem 3.5.9 finish the proof. □

Now we present the main result of this section. The proof of the following compactness result is partly inspired by that of [82, Lemma 3.6]. This result should be compared with [91, Th. 1.2], in which a $W^{s,p}$ version is done. In what follows, given $p \in [1,n)$ we denote by p^* its Sobolev conjugate exponent, i.e., $p^* = \frac{pn}{n-p}$. Recall also the notation p_s^* from Theorem 3.5.14.

Theorem 5.2.2. *Let $1 < p < \infty$ and $g \in W^{1,p}(\mathbb{R}^n)$. For each $s \in (0,1)$, let $u_s \in H_g^{s,p}(\Omega)$ be such that the family $\{D^s u_s\}_{s\in(0,1)}$ is bounded in $L^p(\mathbb{R}^n)$. Then, there exist $u \in W^{1,p}(\mathbb{R}^n)$ and an increasing sequence $\{s_j\}_{j\in\mathbb{N}} \subset (0,1)$ with $\lim_{j\to\infty} s_j = 1$ such that for every q satisfying*

$$\begin{cases} q \in [p,p^*) & \text{if } p < n, \\ q \in [p,\infty) & \text{if } p = n, \\ q \in [p,\infty] & \text{if } p > n, \end{cases}$$

there exists $j_q \in \mathbb{N}$ for which $\{u_{s_j}\}_{j\geq j_q} \subset L^q(\mathbb{R}^n)$ and the convergences

$$u_{s_j} \to u \text{ in } L^q(\mathbb{R}^n) \quad \text{and} \quad D^{s_j}u_{s_j} \rightharpoonup Du \text{ in } L^p(\mathbb{R}^n)$$

hold as $j \to \infty$.

Proof. Thanks to Theorem 5.1.1 and the Sobolev embedding, we can assume, without loss of generality, that $g = 0$.

Fix $0 < \bar{s} < s_0 < 1$. By hypothesis and Proposition 5.2.1, $\{D^{\bar{s}}u_s\}_{s\in[s_0,1)}$ is bounded in $L^p(\mathbb{R}^n,\mathbb{R}^n)$, and consequently, by Corollary 3.5.10, $\{u_s\}_{s\in[s_0,1)}$ is bounded in $H_0^{\bar{s},p}(\Omega)$. Since $H_0^{\bar{s},p}(\Omega)$ is reflexive, there exist $u \in H_0^{\bar{s},p}(\Omega)$ and an increasing sequence $\{s_j\}_{j\geq1} \subset [s_0,1)$, with $\lim_{j\to\infty} s_j = 1$, such that

$$u_{s_j} \rightharpoonup u \quad \text{in } H_0^{\bar{s},p}(\Omega).$$

Now, if $p \leq n$, given $q \in [p, p^*)$ (defining $p^* = \infty$ if $p = n$), there exists $j_q \in \mathbb{N}$ such that for all $j \geq j_0$ we have $q < p^*_{s_j}$. Arguing as above we obtain that $u_{s_j} \rightharpoonup u$ in $H_0^{s_{j_0},p}(\Omega)$, so applying Theorem 3.5.14, we have that $\{u_{s_j}\}_{j \geq j_q} \subset L^q(\mathbb{R}^n)$ and

$$u_{s_j} \to u \quad \text{in } L^q(\mathbb{R}^n).$$

If $p > n$, there exists $j_0 \in \mathbb{N}$ such that $s_{j_0} p > n$, and arguing as above using again Theorem 3.5.14, we have that $\{u_{s_j}\}_{j \geq j_0} \subset L^q(\mathbb{R}^n)$, for any $q \in [p, +\infty]$, and

$$u_{s_j} \to u \quad \text{in } L^q(\mathbb{R}^n).$$

Next, we have that, as $\{D^{s_j} u_{s_j}\}_{j \geq j_0}$ is bounded in $L^p(\mathbb{R}^n, \mathbb{R}^n)$, there exists $V \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that $D^{s_j} u_{s_j} \rightharpoonup V$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $j \to \infty$, in principle up to a subsequence, but we will see that in fact it holds true for the whole sequence. Given $\varphi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, using the fractional integration by parts, Proposition 3.2.4, we get

$$\int D^{s_j} u_{s_j}(x) \cdot \varphi(x)\, dx = -\int u_{s_j}(x) \operatorname{div}^{s_j} \varphi(x)\, dx,$$

and passing to the limit as $j \to \infty$, having in mind that both u_{s_j} and $\operatorname{div}^{s_j} \varphi$ are strongly convergent (Corollary 5.1.2), we obtain

$$\int V(x) \cdot \varphi(x)\, dx = -\int u(x) \operatorname{div} \varphi(x)\, dx,$$

and hence $Du = V$ and $u \in W^{1,p}(\mathbb{R}^n)$. Since this V is unique, this shows that $D^{s_j} u_{s_j} \rightharpoonup V$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $j \to \infty$ without the need of taking a subsequence. This finishes the proof. $\square$

5.3 Weak continuity of the minors for varying s

In this section we prove the analogue in this context of the weak continuity of minors, namely, that if we have a sequence $\{u_s\}_{s \in (0,1)}$ such that $u_s \in H^{s,p}$ for each s and $D^s u_s \rightharpoonup Du$ in L^p as $s \nearrow 1$ for some $u \in W^{1,p}$ then the minors of $D^s u_s$ converge weakly in some L^q to the minors of Du. For this, we follow the general guidelines of Section 4.2, where the analogue convergence for a fixed s is proved.

The following result is the key to adapt the continuity of minors of Theorem 4.2.2 to our case. It establishes the relationship between the operators K^s_φ and $D\varphi$ when $s \nearrow 1$.

Lemma 5.3.1. *Let $p > 1$ and $0 < s < 1$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $w \in L^p(\mathbb{R}^n, \mathbb{R}^{n\times n})$. Consider a family $\{w_s\}_{s\in(0,1)}$ in $L^p(\mathbb{R}^n, \mathbb{R}^{n\times n})$ such that $w_s \rightharpoonup w$ in $L^p(\mathbb{R}^n, \mathbb{R}^{n\times n})$ as $s \nearrow 1$. Then, for all $r \in (1, p]$,*

$$K_\varphi^s(w_s) \rightharpoonup w\, D\varphi \quad \text{in } L^r(\mathbb{R}^n, \mathbb{R}^n) \text{ as } s \nearrow 1.$$

Proof. Assume first that $w_s \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n\times n})$ for all $s \in (0,1)$. Fix two indexes $1 \le i, j \le n$, let u_s be the (i,j)-th entry of w_s and let u be the (i,j)-th entry of w. Let $\theta \in C_c^\infty(\mathbb{R}^n)$. We apply the product formula of Lemma 3.3.6 and then Proposition 3.2.4 to obtain

$$\int \theta\, K_\varphi^s(u_s I) = \int \theta\, D^s(\varphi u_s) - \int \theta\, \varphi\, D^s u_s = -\int \varphi\, u_s \operatorname{div}^s \theta + \int u_s \operatorname{div}^s(\theta\varphi).$$

Now we have from Corollary 5.1.2 that $\operatorname{div}^s \theta \to \operatorname{div} \theta$ and $\operatorname{div}^s(\theta\varphi) \to \operatorname{div}^s(\theta\varphi)$ in $L^q(\mathbb{R}^n)$ for every $q \in (1,\infty)$ as $s \nearrow 1$. As $u_s \rightharpoonup u$ in $L^p(\mathbb{R}^n)$ we obtain

$$\int \theta\, K_\varphi^s(u_s I) \to -\int \varphi\, u \operatorname{div} \theta + \int u \operatorname{div}(\theta\varphi) = \int \theta\, u\, D\varphi.$$

This shows that $K_\varphi^s(u_s I) \rightharpoonup u\, D\varphi$ in the sense of distributions. Now, by Lemma 3.3.2, for every $r \in (1, p]$

$$\left\| K_\varphi^s(u_s I) \right\|_r \le C \left\| u_s \right\|_p \le C_1$$

for some $C, C_1 > 0$ independent of s, which implies that $K_\varphi^s(u_s I) \rightharpoonup u\, D\varphi$ in $L^r(\mathbb{R}^n)$.

Now, we remove the assumption $w_s \in W^{1,p}(\mathbb{R}^n)$. Fix $r \in (1, p]$. For each $s \in (0,1)$, let $v_s \in W^{1,p}(\mathbb{R}^n)$ be such that $\|u_s - v_s\|_r \le 1 - s$. By Lemma 3.3.2,

$$\|K_\varphi^s(u_s I) - K_\varphi^s(v_s I)\|_r = \|K_\varphi^s(u_s I - v_s I)\|_r \le C \|u_s - v_s\|_p \to 0,$$

which implies that $K_\varphi^s(u_s I) \rightharpoonup u\, D\varphi$ in $L^r(\mathbb{R}^n)$. In other words, for each $j \in \{1, \dots, n\}$, the family of functions

$$x \mapsto c_{n,s} \int \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+s}} u_s(y) \frac{x_j - y_j}{|x-y|} dy$$

converges weakly to $u(D\varphi)_j$ as $s \nearrow 1$. Therefore, for each $i \in \{1, \dots, n\}$, the family of functions

$$x \mapsto \left(K_\varphi^s(w_s)\right)_{ij}(x) = \sum_{j=1}^n c_{n,s} \int \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+s}} (w_s)_{ij}(y) \frac{x_j - y_j}{|x-y|} dy$$

converges weakly to $\sum_{j=1}^n w_{ij} (D\varphi)_j = (w\, D\varphi)_i$. This concludes the proof. $\square$

The following is the main result of this section, and shows the weak convergence of the minors of $D^s u_s$ to those of Du, whenever u_s converges weakly to u. Of course, by a minor we mean the determinant of a submatrix. Its proof is an adaptation of Theorem 4.2.2 and we will use again the notation from Definition 4.1.1.

Theorem 5.3.2. *Let $p \geq n-1$ and $0 < s < 1$. Let $g \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ and $u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$. Let $\{u_s\}_{s\in(0,1)}$ be a family such that $u_s \in H^{s,p}_g(\Omega, \mathbb{R}^n)$ for each $s \in (0,1)$, while $u_s \to u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $D^s u_s \rightharpoonup Du$ in $L^p(\mathbb{R}^n, \mathbb{R}^{n\times n})$ as $s \nearrow 1$. Then*

a) If $k \in \mathbb{N}$ with $1 \leq k \leq n-2$ and μ is a minor of order k then $\mu(D^s u_s) \rightharpoonup \mu(Du)$ in $L^{\frac{p}{k}}(\mathbb{R}^n)$ as $s \nearrow 1$.

b) If $\operatorname{cof} D^s u_s \rightharpoonup \vartheta$ in $L^q(\mathbb{R}^n, \mathbb{R}^{n\times n})$ as $s \nearrow 1$ for some $q \in [1, \infty)$ and $\vartheta \in L^q(\mathbb{R}^n, \mathbb{R}^{n\times n})$ then $\vartheta = \operatorname{cof} Du$.

c) Assume $\det D^s u_s \rightharpoonup \theta$ in $L^\ell(\mathbb{R}^n)$ as $s \nearrow 1$ for some $\ell \in [1, \infty)$ and some $\theta \in L^\ell(\mathbb{R}^n)$. If $p < n$ assume, in addition, that $\operatorname{cof} D^s u_s \rightharpoonup \operatorname{cof} Du$ in $L^q(\mathbb{R}^n, \mathbb{R}^{n\times n})$ as $s \nearrow 1$ for some $q \in (\frac{p^*}{p^*-1}, \infty)$. Then $\theta = \det Du$.

Proof. We will prove *a)* by induction on k. For $k = 1$ there is nothing to prove. Assume it holds for some $k \leq n-3$ and let us prove it for $k+1$. Let μ be a minor of order $k+1$. In the notation of Definition 4.1.1, $\mu(F) = \det[F]_M$ for all $F \in \mathbb{R}^{n\times n}$, where $[\cdot]_M = [\cdot]_{M_{i_1,\ldots,i_{k+1};j_1,\ldots,j_{k+1}}}$ for some $1 \leq i_1 < \cdots < i_{k+1} \leq n$ and $1 \leq j_1 < \cdots < j_{k+1} \leq n$. Let $\varphi \in C^\infty_c(\mathbb{R}^n)$. By induction assumption, $\operatorname{cof}[D^s u_s]_M \rightharpoonup \operatorname{cof}[Du]_M$ in $L^{\frac{p}{k}}(\mathbb{R}^n, \mathbb{R}^{(k+1)\times(k+1)})$ as $s \nearrow 1$, so $[\operatorname{cof}[D^s u_s]_M]_{\bar{M}} \rightharpoonup [\operatorname{cof}[Du]_M]_{\bar{M}}$ in $L^{\frac{p}{k}}(\mathbb{R}^n, \mathbb{R}^{n\times n})$. By Lemma 5.3.1, $K^s_\varphi([\operatorname{cof}[D^s u_s]_M]_{\bar{M}}) \rightharpoonup [\operatorname{cof}[Du]_M]_{\bar{M}}\, D\varphi$ in $L^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $r \in (1, \frac{p}{k}]$. By Theorem 5.2.2, $[u_s]_{\tilde{N}} \to [u]_{\tilde{N}}$ in $L^p(\mathbb{R}^n)$, so

$$[u_s]_{\tilde{N}} \cdot K^s_\varphi([\operatorname{cof}[D^s u_s]_M]_{\bar{M}}) \rightharpoonup [u]_{\tilde{N}} \cdot ([\operatorname{cof}[Du]_M]_{\bar{M}}\, D\varphi) \quad \text{in } L^1(\mathbb{R}^n) \tag{5.12}$$

since $\frac{k}{p} + \frac{1}{p} \leq 1$. Now, the nonlocal integration by parts for the determinant given in Lemma 4.2.1 as well as the classical (local) one state that

$$-\frac{1}{k}\int [u_s]_{\tilde{N}}(x) \cdot K^s_\varphi([\operatorname{cof}[D^s u_s]_M]_{\bar{M}})(x)\, dx = \int \det[D^s u_s(x)]_M\, \varphi(x)\, dx \tag{5.13}$$

and

$$-\frac{1}{k}\int [u]_{\tilde{N}}(x) \cdot ([\operatorname{cof}[Du]_M]_{\bar{M}}(x)\, D\varphi(x))\, dx = \int \det[Du(x)]_M\, \varphi(x)\, dx, \tag{5.14}$$

respectively, so

$$\int \det[D^s u_s(x)]_M \, \varphi(x)\, dx \to \int \det[Du(x)]_M \, \varphi(x)\, dx. \tag{5.15}$$

This shows that $\det[D^s u_s]_M \rightharpoonup \det[Du]_M$ in the sense of distributions. As $\{\det[D^s u_s]_M\}_{s\in(0,1)}$ is bounded in $L^{\frac{p}{k+1}}(\mathbb{R}^n)$ and $p > k+1$, we have that $\det[D^s u_s]_M \rightharpoonup \det[Du]_M$ in $L^{\frac{p}{k+1}}(\mathbb{R}^n)$ as $s \nearrow 1$.

The proof of *b)* follows the lines of *a)*. Let μ be a minor of order $n-1$. As before, $\mu(F) = \det[F]_M$ for all $F \in \mathbb{R}^{n\times n}$, where $[\cdot]_M = [\cdot]_{M_{i_1,\dots,i_{n-1};j_1,\dots,j_{n-1}}}$ for some $1 \le i_1 < \cdots < i_{n-1} \le n$ and $1 \le j_1 < \cdots < j_{n-1} \le n$. Let $\phi \in C^\infty_c(\Omega)$. By part *a)*, $\operatorname{cof}[D^s u_j]_M \rightharpoonup \operatorname{cof}[D^s u]_M$ in $L^{\frac{p}{n-2}}(\mathbb{R}^n, \mathbb{R}^{(n-1)\times(n-1)})$, so $[\operatorname{cof}[D^s u_s]_M]_{\bar{M}} \rightharpoonup [\operatorname{cof}[Du]_M]_{\bar{M}}$ in $L^{\frac{p}{n-2}}(\mathbb{R}^n, \mathbb{R}^{n\times n})$. By Lemma 5.3.1, $K^s_\varphi([\operatorname{cof}[D^s u_s]_M]_{\bar{M}}) \rightharpoonup [\operatorname{cof}[Du]_M]_{\bar{M}}\, D\varphi$ in $L^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $r \in (1, \frac{p}{n-2}]$. By Theorem 5.2.2, $[u_s]_{\tilde{N}} \to [u]_{\tilde{N}}$ in $L^p(\mathbb{R}^n)$, so convergence (5.12) is also valid since $\frac{n-2}{p} + \frac{1}{p} \le 1$. Again thanks to (5.13)–(5.14), we conclude that convergence (5.15) holds. This shows that $\mu(D^s u_s) \rightharpoonup \mu(Du)$ in the sense of distributions. As this is true for every minor μ of order $n-1$, we obtain that $\operatorname{cof} D^s u_s \rightharpoonup \operatorname{cof} Du$ in the sense of distributions. Due to the assumption, $\vartheta = \operatorname{cof} Du$.

We finally show part *c)*. Let $\phi \in C^\infty_c(\Omega)$. Assume first $p < n$. By the assumption and Lemma 5.3.1, $K^s_\varphi(\operatorname{cof} D^s u_s) \rightharpoonup \operatorname{cof} Du\, D\varphi$ in $L^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $r \in (1, q]$. By Theorem 5.2.2, $u_s \to u$ in $L^t(\mathbb{R}^n)$ for every $t \in [p, p^*)$, so

$$u_s \cdot K^s_\varphi(\operatorname{cof} D^s u_s) \rightharpoonup u \cdot (\operatorname{cof} Du\, D\varphi) \quad \text{in } L^1(\mathbb{R}^n) \tag{5.16}$$

since $\frac{1}{q} + \frac{1}{p^*} < 1$.

Assume now $p \ge n$. Then $\{\operatorname{cof} D^s u_s\}_{s\in(0,1)}$ is bounded in $L^{\frac{p}{n-1}}(\mathbb{R}^n, \mathbb{R}^{n\times n})$ so, thanks to part *b)*, $\operatorname{cof} D^s u_s \rightharpoonup \operatorname{cof} Du$ in $L^{\frac{p}{n-1}}(\mathbb{R}^n, \mathbb{R}^{n\times n})$. By Lemma 5.3.1, $K^s_\varphi(\operatorname{cof} D^s u_s) \rightharpoonup \operatorname{cof} Du\, D\varphi$ in $L^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $r \in (1, \frac{p}{n-1}]$. By Theorem 5.2.2, $u_s \to u$ in $L^t(\mathbb{R}^n)$ for every $t \in [1, \infty)$, so convergence (4.31) holds since $p > n-1$.

In either case, we have convergence (5.16), so by the analogue of (5.13)–(5.14) with $k = n$ we obtain

$$\int \det D^s u_s(x)\, \varphi(x)\, dx \to \int \det Du(x)\, \varphi(x)\, dx.$$

This shows that $\det D^s u_s \rightharpoonup \det Du$ in the sense of distributions, so $\theta = \det Du$. □

5.4 Γ-convergence

Γ-convergence is the main conceptual tool for studying the variational convergence of families of functionals defined on metric spaces [29]. In this section we show that the functional

$$\mathcal{I}_s(u) = \int W(x, u(x), D^s u(x))\, dx,$$

defined on $H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$, Γ-converges, as $s \nearrow 1$, to the functional

$$\mathcal{I}(u) = \int W(x, u(x), Du(x))\, dx,$$

defined on $W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ under the assumption of W being polyconvex. We recall (again, see Section 4.3)the concept of polyconvexity (see, e.g, [10, 39]). Let τ be the number of submatrices of an $n \times n$ matrix. We fix a function $\vec{\mu} : \mathbb{R}^{n\times n} \to \mathbb{R}^\tau$ such that $\vec{\mu}(F)$ is the collection of all minors of an $F \in \mathbb{R}^{n\times n}$ in a given order. A function $W_0 : \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ is polyconvex if there exists a convex $\Phi : \mathbb{R}^\tau \to \mathbb{R} \cup \{\infty\}$ such that $W_0(F) = \Phi(\vec{\mu}(F))$ for all $F \in \mathbb{R}^{n\times n}$. Polyconvexity of $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ means polyconvexity in the last variable.

It is convenient to consider both $\mathcal{I}_s$ and $\mathcal{I}$ defined on the same functional space independent of s, so we consider both functionals defined on $L^p(\mathbb{R}^n, \mathbb{R}^n)$. The extension of $\mathcal{I}_s$ to $L^p(\mathbb{R}^n, \mathbb{R}^n) \setminus H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ and of $\mathcal{I}$ to $L^p(\mathbb{R}^n, \mathbb{R}^n) \setminus W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ is done by infinity. Recalling the definition of Γ-convergence in this particular situation, we say that $\mathcal{I}_s$ Γ-converges to $\mathcal{I}$ as $s \nearrow 1$ in the strong topology of $L^p(\mathbb{R}^n, \mathbb{R}^n)$ if the following two conditions hold:

- *Liminf inequality:* For every family $\{u_s\}_{s\in(0,1)}$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that $u_s \to u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $s \nearrow 1$, we have

$$\mathcal{I}(u) \leq \liminf_{s\nearrow 1} \mathcal{I}_s(u_s).$$

- *Limsup inequality:* For each $u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$, there exists a family $\{u_s\}_{s\in(0,1)} \subset L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that $u_s \to u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $s \nearrow 1$ and

$$\limsup_{s\nearrow 1} \mathcal{I}_s(u_s) \leq \mathcal{I}(u).$$

Although not in the definition of Γ-convergence, it is customary to attach a compactness property to the conditions above, which, in this context, reads as follows:

- *Compactness:* For every $g \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ and every family $\{u_s\}_{s\in(0,1)}$ with $u_s = g$ in Ω^c for all $s \in (0,1)$ such that $\liminf_{s\nearrow 1} \mathcal{I}_s(u_s) < \infty$, there exist an increasing sequence $\{s_j\}_{j\in\mathbb{N}}$ in $(0,1)$ with $\lim_{j\to\infty} s_j = 1$ and a $u \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that $u_{s_j} \to u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $j \to \infty$.

The *limsup inequality* will be a consequence of Theorem 5.1.1, while the compactness property will follow of Theorem 5.2.2. The *liminf inequality*, on the other hand, is a novel semicontinuity result, which improves that of Theorem 4.3.1 done for a fixed s, and is singled out in the following proposition. As we will see, the growth conditions for proving the *liminf* and *limsup* inequalities are compatible only in the range $p > n$.

Proposition 5.4.1. *Let $p \geq n-1$ satisfy $p > 1$ and $0 < s < 1$. Let Ω be a bounded open subset of $\mathbb{R}^n$ and $g \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$. Let $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ satisfy the following conditions:*

a) W is $\mathcal{L}^n \times \mathcal{B}^n \times \mathcal{B}^{n\times n}$-measurable, where $\mathcal{L}^n$ denotes the Lebesgue sigma-algebra in $\mathbb{R}^n$, whereas $\mathcal{B}^n$ and $\mathcal{B}^{n\times n}$ denote the Borel sigma-algebras in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, respectively.

b) $W(x,\cdot,\cdot)$ is lower semicontinuous for a.e. $x \in \mathbb{R}^n$.

c) For a.e. $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}^n$, the function $W(x,y,\cdot)$ is polyconvex.

d) There exist a constant $c > 0$, an $a \in L^1(\mathbb{R}^n)$ and a Borel function $h : [0,\infty) \to [0,\infty)$ such that

$$\lim_{t\to\infty} \frac{h(t)}{t} = \infty$$

and, for some $q > \frac{p^*}{p^*-1}$ if $p < n$,

$$\begin{cases} W(x,y,F) \geq a(x) + c\,|F|^p + c\,|\mathrm{cof}\, F|^q + h(|\det F|), & \text{if } p < n, \\ W(x,y,F) \geq a(x) + c\,|F|^p + h(|\det F|), & \text{if } p = n, \\ W(x,y,F) \geq a(x) + c\,|F|^p\,, & \text{if } p > n, \end{cases}$$

for a.e. $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$ and all $F \in \mathbb{R}^{n\times n}$.

For each $s \in (0,1)$, let $u_s \in H^{s,p}_g(\Omega, \mathbb{R}^n)$ and $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ satisfy $u_s \to u$ in $L^p(\Omega, \mathbb{R}^n)$ as $s \nearrow 1$. Then

$$\mathcal{I}(u) \leq \liminf_{s\nearrow 1} \mathcal{I}_s(u_s) \tag{5.17}$$

Proof. We can assume that

$$\liminf_{s\nearrow 1} \mathcal{I}_s(u_s) < \infty, \tag{5.18}$$

hence by assumption *d)*, there exists an increasing sequence $\{s_j\}_{j\in\mathbb{N}}$ in $(0,1)$ with $\lim_{j\to\infty} s_j = 1$ such that $\liminf_{s\nearrow 1} \mathcal{I}_s(u_s) = \lim_{j\to\infty} \mathcal{I}_{s_j}(u_{s_j})$ and the sequence $\{D^{s_j}u_{s_j}\}_{j\in\mathbb{N}}$ is bounded in $L^p(\mathbb{R}^n,\mathbb{R}^n)$, so by Theorem 5.2.2, for a subsequence (not relabelled),

$$u_{s_j} \to u \quad \text{and} \quad D^{s_j}u_{s_j} \rightharpoonup Du \qquad \text{in } L^p \text{ as } j\to\infty. \tag{5.19}$$

By Theorem 4.2.2, for any minor μ of order $k \le n-2$, we have that

$$\mu(D^{s_j}u_{s_j}) \rightharpoonup \mu(Du) \text{ in } L^{\frac{p}{k}}(\mathbb{R}^n) \text{ as } j\to\infty. \tag{5.20}$$

If $p<n$ then, $\{\operatorname{cof} D^s u_s\}_{0<s<1}$ is bounded in $L^q(\mathbb{R}^n,\mathbb{R}^{n\times n})$ by assumption *d)*, whereas if $p\ge n$ we set $q := \frac{p}{n-1}$ and have that $\{\operatorname{cof} D^s u_s\}_{0<s<1}$ is bounded in $L^q(\mathbb{R}^n,\mathbb{R}^{n\times n})$. In either case we have that $q>1$, so for a subsequence $\{\operatorname{cof} D^{s_j}u_{s_j}\}_{j\in\mathbb{N}}$ converges weakly in $L^q(\mathbb{R}^n,\mathbb{R}^{n\times n})$ and, by Theorem 4.2.2,

$$\operatorname{cof} D^{s_j}u_{s_j} \rightharpoonup \operatorname{cof} Du \text{ in } L^q(\mathbb{R}^n,\mathbb{R}^{n\times n}) \text{ as } j\to\infty. \tag{5.21}$$

If $p\le n$ then, by assumption *d)* and de la Vallée Poussin's criterion, $\{\det D^s u_s\}_{0<s<1}$ is equiintegrable, whereas if $p>n$, $\{\det D^s u_s\}_{0<s<1}$ is bounded in $L^{\frac{p}{n}}(\mathbb{R}^n)$ and $\frac{p}{n}>1$. In either case we have that, for a subsequence, $\{\det D^{s_j}u_{s_j}\}_{j\in\mathbb{N}}$ converges weakly in $L^\ell(\mathbb{R}^n)$ with

$$\begin{cases} \ell = 1 & \text{if } p\le n,\\ \ell = \frac{p}{n} & \text{if } p>n, \end{cases}$$

and, hence, by Theorem 4.2.2,

$$\det D^{s_j}u_{s_j} \rightharpoonup \det Du \text{ in } L^\ell(\mathbb{R}^n) \text{ as } j\to\infty. \tag{5.22}$$

Convergences (5.19)–(5.22) imply, thanks to a standard lower semicontinuity result for polyconvex functionals (see, e.g., [15, Th. 5.4] or [59, Th. 7.5]), that for any $R>0$,

$$\int_{B(0,R)} W(x,u(x),Du(x))\,dx \le \liminf_{j\to\infty} \int_{B(0,R)} W(x,u_{s_j}(x),D^{s_j}u_{s_j}(x))\,dx. \tag{5.23}$$

Therefore,

$$\int_{B(0,R)} [W(x,u(x),Du(x)) - a(x)]\,dx \leq$$
$$\liminf_{j\to\infty} \int \left[W(x,u_{s_j}(x),D^{s_j}u_{s_j}(x)) - a(x)\right] dx.$$

By monotone convergence,

$$\int [W(x,u(x),Du(x)) - a(x)]\,dx \leq$$
$$\liminf_{j\to\infty} \int \left[W(x,u_{s_j}(x),D^{s_j}u_{s_j}(x)) - a(x)\right] dx,$$

so

$$\mathcal{I}(u) \leq \liminf_{j\to\infty} \mathcal{I}_{s_j}(u_{s_j}),$$

as desired. □

We finally present the main result of this section, which shows the Γ-convergence of polyconvex functionals defined on Bessel spaces, involving s-fractional gradients, to a classical local polyconvex functional defined on a Sobolev space. Unfortunately, we crucially need the extra assumption $p > n$ in order to prove the *limsup inequality*. This is because the coercivity conditions of W in Proposition 5.4.1 are compatible with the standard upper bound by $|F|^p$ (which makes the functional $\mathcal{I}$ continuous in $W^{1,p}$; see [39]) only in the case $p > n$.

Theorem 5.4.2. *Let $p > n$ and $0 < s < 1$. Let Ω be a bounded open subset of $\mathbb{R}^n$ and $g \in W^{1,p}(\mathbb{R}^n,\mathbb{R}^n)$. Let $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ satisfy the following conditions:*

a) W is $\mathcal{L}^n \times \mathcal{B}^n \times \mathcal{B}^{n\times n}$-measurable.

b) $W(x,\cdot,\cdot)$ is lower semicontinuous for a.e. $x \in \mathbb{R}^n$.

c) For a.e. $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}^n$, the function $W(x,y,\cdot)$ is polyconvex.

d) Assume there exist $c > 0$ and $a \in L^1(\mathbb{R}^n)$ such that

$$W(x,y,F) \geq a(x) + c\,|F|^p, \qquad \text{a.e. } x \in \mathbb{R}^n, \text{ all } y \in \mathbb{R}^n, \text{ all } F \in \mathbb{R}^{n\times n},$$

and for every $R > 0$ there exist $a_R \in L^1(\mathbb{R}^n)$ and $c_R > 0$ such that for a.e. $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$ with $|y| \leq R$ and all $F \in \mathbb{R}^{n\times n}$,

$$W(x,y,F) \leq a_R(x) + c_R\,|F|^p\,.$$

The following statements hold:

i) For each $s \in (0,1)$, let $u_s \in H^{s,p}_g(\Omega, \mathbb{R}^n)$ satisfy

$$\liminf_{s \nearrow 1} \mathcal{I}_s(u_s) < \infty. \tag{5.24}$$

Then there exist $u \in W^{1,p}_g(\Omega, \mathbb{R}^n)$ and an increasing sequence $\{s_j\}_{j\in\mathbb{N}}$ in $(0,1)$ with $\lim_{j\to\infty} s_j = 1$ such that $u_{s_j} \to u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $j \to \infty$.

ii) For each $s \in (0,1)$, let $u_s \in H^{s,p}_g(\Omega, \mathbb{R}^n)$ and $u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ satisfy $u_s \to u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$. Then

$$\mathcal{I}(u) \leq \liminf_{s \nearrow 1} \mathcal{I}_s(u_s).$$

iii) For each $u \in W^{1,p}_g(\Omega, \mathbb{R}^n)$ and $s \in (0,1)$, there exists $u_s \in H^{s,p}_g(\Omega, \mathbb{R}^n)$ such that $u_s \to u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ and

$$\limsup_{s \nearrow 1} \mathcal{I}_s(u_s) \leq \mathcal{I}(u). \tag{5.25}$$

Proof. For proving *i)*, just notice that by assumption *d)*, (5.24) implies that there is an increasing sequence $\{s_j\}_{j\in\mathbb{N}}$ in $(0,1)$ with $\lim_{j\to\infty} s_j = 1$ such that $\{D^{s_j} u_{s_j}\}_{j\in\mathbb{N}}$ is bounded in $L^p(\mathbb{R}^n, \mathbb{R}^n)$. Therefore, by Theorem 5.2.2, there exists $u \in W^{1,p}_g(\Omega, \mathbb{R}^n)$ such that, for a subsequence $u_{s_j} \to u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $j \to \infty$.

Part *ii)* is a particular case of Proposition 5.4.1.

Finally we show *iii)*, so we let $u \in W^{1,p}_g(\Omega, \mathbb{R}^n)$. By Theorem 5.1.1, $D^s u \to Du$ in $L^p(\mathbb{R}^n, \mathbb{R}^{n\times n})$ as $s \nearrow 1$. Assumption *c)* implies in particular the continuity of $W(x, y, \cdot)$ for a.e. $x \in \mathbb{R}^n$ and all $y \in \mathbb{R}^n$ (see, e.g., [39]). The Sobolev embedding shows that u is bounded. By the growth conditions and dominated convergence,

$$\lim_{s \nearrow 1} \int W(x, u, D^s u) = \int W(x, u, Du), \tag{5.26}$$

which proves (5.25). $\square$

Although the bulk of this section has been focused on the assumption of polyconvexity, with the stronger assumption of convexity we can achieve the analogue result of Theorem 5.4.2 for the full range of exponents $p \in (1, \infty)$. Since the proof is analogous (and in some steps, simpler) than that of Theorem 5.4.2, it will only be sketched.

Theorem 5.4.3. *Let $1 < p < \infty$ and $0 < s < 1$. Let Ω be a bounded open subset of $\mathbb{R}^n$ and $g \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$. Let $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ satisfy the following conditions:*

a) W is $\mathcal{L}^n \times \mathcal{B}^n \times \mathcal{B}^{n\times n}$-measurable.

b) $W(x, \cdot, \cdot)$ is lower semicontinuous for a.e. $x \in \mathbb{R}^n$.

c) For a.e. $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}^n$, the function $W(x, y, \cdot)$ is convex.

d1) If $p < n$ assume there exist $c \geq 1$ and $a \in L^1(\mathbb{R}^n)$ such that for a.e. $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$ and all $F \in \mathbb{R}^{n\times n}$,

$$-a(x) + \frac{1}{c}\,|F|^p \leq W(x, y, F) \leq a(x) + c\left(|y|^p + |y|^{p^*} + |F|^p\right).$$

d2) If $p = n$ assume there exist $r \in [p, \infty)$, $c \geq 1$ and $a \in L^1(\mathbb{R}^n)$ such that for a.e. $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$ and all $F \in \mathbb{R}^{n\times n}$,

$$-a(x) + \frac{1}{c}\,|F|^p \leq W(x, y, F) \leq a(x) + c\left(|y|^p + |y|^r + |F|^p\right).$$

d3) If $p > n$ assume there exist $c > 0$ and $a \in L^1(\mathbb{R}^n)$ such that

$$W(x, y, F) \geq a(x) + c\,|F|^p, \qquad \text{a.e. } x \in \mathbb{R}^n, \text{ all } y \in \mathbb{R}^n, \text{ all } F \in \mathbb{R}^{n\times n},$$

and for every $R > 0$ there exist $a_R \in L^1(\mathbb{R}^n)$ and $c_R > 0$ such that for a.e. $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$ with $|y| \leq R$ and all $F \in \mathbb{R}^{n\times n}$,

$$W(x, y, F) \leq a_R(x) + c_R\,|F|^p.$$

Then, statements i)–iii) of Theorem 5.4.2 hold.

Proof. The proof of *i)* is the same as that of Theorem 5.4.2.

For the proof of *ii)* we initially follow that of Proposition 5.4.1. We can assume inequality (5.18), so there exists an increasing sequence $\{s_j\}_{j\in\mathbb{N}}$ in $(0, 1)$ with $\lim_{j\to\infty} s_j = 1$ such that $\liminf_{s\nearrow 1} \mathcal{I}_s(u_s) = \lim_{j\to\infty} \mathcal{I}_{s_j}(u_{s_j})$ and the sequence $\{D^{s_j} u_{s_j}\}_{j\in\mathbb{N}}$ is bounded in $L^p(\mathbb{R}^n, \mathbb{R}^n)$. By Theorem 5.2.2, for a subsequence, convergences (5.19) hold. By a standard lower semicontinuity result for convex functionals (see, e.g., [59, Th. 7.5]), for any $R > 0$, inequality (5.23) holds, and we conclude (5.17) as in Proposition 5.4.1.

In order to show *iii)* we apply Theorem 5.1.1 and obtain $D^s u \to Du$ in $L^p(\mathbb{R}^n, \mathbb{R}^{n\times n})$ as $s \nearrow 1$. Assumption *c)* implies in particular the continuity of $W(x, y, \cdot)$ for a.e. $x \in \mathbb{R}^n$ and all $y \in \mathbb{R}^n$. The Sobolev embedding in the three cases ($p < n$, $p = n$ and $p > n$) shows that the growth conditions allow us to apply dominated convergence and conclude inequality (5.26), as desired. □

Part III

A nonlocal model of hyperelasticity

Chapter 6. Nonlocal framework on bounded domains

As a quick recapitulation, we have commented on the increased prominence of nonlocal models in the last decades, for which it is required a thorough mathematical analysis of the new objects and operators involved. We have already analysed some of them in this document. In Part I we have studied the localization of the nonlinear bond-based model of peridynamics. Then, in Part II we have considered the study of fractional spaces and functionals based on the s-fractional gradient, an operator that generalizes the classical gradient to a degree of differentiability beyond derivatives of natural order. Now, this Part III, which encompasses the results from the *in preparation* articles [18,19], intends to fix the following drawbacks of some nonlocal models in Solid Mechanics. First, in Part I it was showed that the model with energy functional

$$I(u) = \int_\Omega \int_{\Omega \cap B(x,\delta)} w(x-y, u(x)-u(y))\, dydx$$

does not fit in nonlinear Solid Mechanics. Secondly, in Part II we analysed fractional variational problems substituting the classical gradient by the one from Definition 3.1.2:

$$\mathcal{I}(u) = \int_{\mathbb{R}^n} W(x, u(x), D^s u(x))\, dx.$$

In these functionals, contrary to hyperelastic modelling, the fractional gradient D^s and the energy functional $\mathcal{I}$ are defined over the whole space $\mathbb{R}^n$. This presents a drawback regarding Solid Mechanics, where it is required to work with bounded domains. Furthermore, this restricts substantially the possibility of working with a wider range of boundary conditions.

Therefore, with these observations in mind and so as to approach the study of a proper model in Solid Mechanics, we would like to study variational models based on an alternative differential operator. In particular, we will work with

$$D_\delta^s u(x) = c_{n,s} \int_{B(x,\delta)} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{w(|x-y|)}{|x-y|^{n+s-1}} \, dy, \tag{6.1}$$

for $u \in C_c^\infty(\mathbb{R}^n)$, where the nonlocal kernel considered is the Riesz potential multiplied by a cut-off function w, so it keeps the index s of the fractional differentiability, but it also gives rise to an operator acting over bounded domains in contrast to the fractional gradient (3.7). Similar operators have already been studied in works like [82, 84], where (6.1) could fit after normalizing its kernel. It has also been shown in [84] that this kind of operators also converges to the classical gradient when the nonlocality vanishes. Thus, among all the properties mentioned at the introduction of Chapter 3 in order to characterise the fractional gradient, the one that is not fulfilled by (6.1) is the s-homogeneity under dilations, in favour of considering bounded domains.

In our case, we have decided to fix as support for such kernel the ball $B(0, \delta)$ of radius $\delta > 0$ (a small distance), since our main motivation comes from Peridynamics, where interaction between particles is assumed to be negligible when they are further away than a certain distance δ, known as the *horizon* distance of interaction of particles. Peridynamics is a nonlocal alternative model in Solid Mechanics proposed by Silling in [103, 107] (see Section 1.3), whose goal is to unite in one model elastic deformations as well as singularity phenomena such as fractures. Although the development of this theory in the last years has been impressive, most of the work until now is on linear elastic models [82, 83]. A first attempt to rigorously extend this nonlocal theory for a general nonlocal nonlinear model was made in [23–25], but in [20] (Part I) was shown that it did not meet its modelling goal in Solid Mechanics.

All these considerations will imply a slight redefinition of the functional space in this nonlocal framework, as well as another look to certain properties such as integration by parts [21, 84] or the search of new Poincaré–Sobolev inequality and compact embedding results.

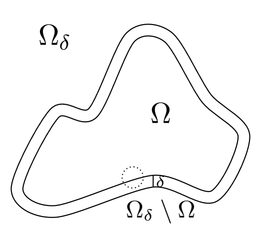
The outline of the chapter is the following. In Section 6.1 the new versions of nonlocal gradient and functional space are established as well as the corresponding version of the nonlocal integration by parts. Then, in Section 6.2 several formulas are computed, such as the nonlocal derivatives of a product, the symmetry of the second derivative, or a relevant linear operator $K_\varphi^{s,\delta}$ similar to the one that appeared in Part II. Section 6.3 leads to the process of obtaining a nonlocal version of the Fundamental Theorem of Calculus, a key and relevant ingredient for the following results. Then in Section 6.4 it is proved a new nonlocal version of the Poincaré–Sobolev inequality, which, along with a *nonlocal mean value theorem*, give rise to the compact embedding result. At the end of this chapter some comments regarding the fractional and nonlocal gradients are shown as well as different notions for a nonlocal laplacian.

In the next and last chapter, the existence of minimizers of nonlocal energy functionals is shown. First, it is done in the scalar case under the convexity assumption. Then the Euler-Lagrane equation are shown. Finally we prove, using the nonlocal Piola identity, the existence of minimizers of vector variational problems under polyconvexity (weaker the convexity), a result which is compatible with the existence of function exhibiting singularities of fracture and cavitation type.

6.1 Functional analysis framework

In this section we state the definitions and basic properties of the nonlocal gradient and divergence, as well as the natural functional space associated to them.

The framework is the following. As typical in nonlocal models, 'boundary' conditions are usually of volumetric type. In our case, we fix a distance $\delta > 0$ and consider a bounded domain $\Omega \subset \mathbb{R}^n$. The set Ω itself is regarded as a nonlocal interior domain, while $\Omega_\delta := \Omega + B(0, \delta)$ is considered as its nonlocal closure. Accordingly, the set $\Omega_{B,\delta} := \Omega_\delta \backslash \Omega$ plays the role of nonlocal boundary. We write $B(x, r)$ to denote the open ball centred at x of radius r.



The set $\Omega_{-\delta} = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta\}$ will also be relevant.

Similarly as in [84], the nonlocal operators are based on an integral kernel. As mentioned in the introduction, we will consider

$$\rho_\delta(\tilde{x}) = \frac{1}{\gamma(1-s)|\tilde{x}|^{n+s-1}} w_\delta(\tilde{x}),$$

with $0 < s < 1$, where the constant $\gamma(s)$ is given by

$$\gamma(s) = \frac{\pi^{\frac{n}{2}} 2^s \Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{n-s}{2}\right)} \tag{6.2}$$

and Γ is Euler's gamma function. We assume the following conditions over the cut-off function w_δ:

a) Smoothness: w_δ is a non negative radial function such that $w_\delta \in C_c^\infty(B(0,\delta))$.

b) Cut-off: There are constants $a_0 > 0$ and $0 < b_0 < 1$ such that $0 \leq w_\delta \leq a_0$ with $w_\delta = a_0$ in $B(0, b_0\delta)$ and $w_\delta = 0$ in $B(0,\delta)^c$.

c) $w_\delta(x) \geq w_\delta(y)$ if $|x| \leq |y|$.

d) $\int_{B(0,\delta)} \rho_\delta(\tilde{x})\, d\tilde{x} = 1$.

The radial representation of w_δ is denoted by $\bar{w}_\delta$.

After this introduction we proceed by setting the definition of principal value. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, then given a function $f : \Omega \to \mathbb{R}$ and $x \in \Omega$ such that $f \in L^1(\Omega \setminus B(x,r))$ for every $r > 0$, the principal value centred at x of $\int_\Omega f$, denoted by

$$\mathrm{pv}_x \int_\Omega f$$

is defined as

$$\lim_{r \to 0} \int_{\Omega \setminus B(x,r)} f,$$

whenever this limit exists.

As we did in the fractional case (see Part II), in order to avoid the principal value, we first establish the definition of nonlocal gradient for smooth functions. The definitions of the nonlocal gradient and divergence are the following. We also recall the definition of the constant $c_{n,s}$ appearing in Definition 3.1.1.

Definition 6.1.1. *Let $0 < s < 1$, $0 < \delta$ and set*

$$c_{n,s} := \frac{n+s-1}{\gamma(1-s)}.$$

a) Let $u \in C_c^\infty(\mathbb{R}^n)$. Then, for every $x \in \Omega$ the nonlocal gradient $D_\delta^s u$ is defined as

$$D_\delta^s u(x) = c_{n,s} \int_{B(x,\delta)} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|^{n+s-1}}\, dy. \tag{6.3}$$

b) Let $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. The nonlocal divergence is defined, for $x \in \Omega$, as

$$\operatorname{div}_\delta^s \phi(x) = -\operatorname{pv}_x c_{n,s} \int_{B(x,\delta)} \frac{\phi(x)+\phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|^{n+s-1}}\, dy.$$

Notice that the integral in (6.3) is absolutely convergent. On the other hand, by odd symmetry,

$$\begin{aligned} -\operatorname{pv}_x \int_{B(x,\delta)} \frac{\phi(x)+\phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|^{n+s-1}}\, dy = \\ \int_{B(x,\delta)} \frac{\phi(x)-\phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|^{n+s-1}}\, dy \end{aligned} \tag{6.4}$$

and this last integral is absolutely convergent.

Note also that, for each $x \in \Omega$,

$$\begin{aligned} \int_{B(x,\delta)} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|^{n+s-1}}\, dy = \\ \int_{\Omega_\delta} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|^{n+s-1}}\, dy, \end{aligned}$$

and similarly for the integral in (6.4), since $B(x,\delta) \subset \Omega_\delta$ and $\operatorname{supp} w_\delta \subset B(0,\delta)$.

Definition 6.1.1 a) naturally extends to vector fields as shown in [84]. Given $u \in C_c^\infty(\Omega_\delta, \mathbb{R}^m)$ measurable and the same setting of Definition 6.1.1, its nonlocal gradient is

$$D_\delta^s u(x) = c_{n,s} \int_{B(x,\delta)} \frac{u(x)-u(y)}{|x-y|} \otimes \frac{x-y}{|x-y|} \frac{w_\delta(|x-y|)}{|x-y|^{n+s-1}}\, dy \quad \forall x \in \Omega, \tag{6.5}$$

Here, $\otimes$ stands for the usual tensor product of vectors.

Analogously, if $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is such that its rows satisfy the assumptions of Definition 3.1.1, we denote by $\operatorname{Div}_\delta^s M$ the column vector-function whose components are the s-nonlocal divergences of each row of M.

The operators of Definition 6.1.1 act as dual operators an integration by parts. Many earlier versions of a nonlocal integration by parts have been proved in different contexts. For the purposes of this work, we will use a particular case of [84, Th. 1.4], which, for convenience, we restate here in our language.

Theorem 6.1.1. *Assume $u \in C_c^\infty(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Then*

$$\int_\Omega \int_\Omega \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|} \cdot \phi(x) \rho_\delta(x-y)\, dy\, dx =$$
$$\int_\Omega u(x)\, \mathrm{pv}_x \int_\Omega \frac{\phi(x) + \phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y)\, dy\, dx.$$

The integration by parts formula suitable in our context is the following.

Theorem 6.1.2. *Let $0 < s < 1$ and $0 < \delta$. Suppose that $u \in C_c^\infty(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\Omega, \mathbb{R}^n)$. Then $D_\delta^s u \in L^\infty(\Omega, \mathbb{R}^n)$ and $\operatorname{div}_\delta^s \phi \in L^\infty(\Omega)$. Moreover,*

$$\int_\Omega D_\delta^s u(x) \cdot \phi(x)\, dx =$$
$$- \int_\Omega u(x) \operatorname{div}_\delta^s \phi(x)\, dx - (n+s-1) \int_\Omega \int_{\Omega_{B,\delta}} \frac{u(y)\phi(x)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y)\, dy\, dx.$$

Proof. Denoting by $L > 0$ the Lipschitz constant of u, we have, for each $x \in \Omega$,

$$|D_\delta^s u(x)| \le c_{n,s} L \int_{B(x,\delta)} \frac{w_\delta(x-y)}{|x-y|^{n+s-1}}\, dy = (n+s-1)L,$$

so $D_\delta^s u \in L^\infty(\Omega, \mathbb{R}^n)$. Analogously, the integral of the right-hand side of (6.4) is absolutely convergent and $\operatorname{div}_\delta^s \phi \in L^\infty(\Omega)$.

We have

$$\int_\Omega D_\delta^s u(x) \cdot \phi(x)\, dx = (n+s-1) \int_\Omega \int_{\Omega_\delta} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \rho_\delta(x-y) \cdot \phi(x)\, dy\, dx$$

with

$$\int_\Omega \int_{\Omega_\delta} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \rho_\delta(x-y) \cdot \phi(x)\, dy\, dx =$$
$$\int_\Omega \int_\Omega \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \rho_\delta(x-y) \cdot \phi(x)\, dy\, dx \ +$$
$$\int_\Omega \int_{\Omega_{B,\delta}} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \rho_\delta(x-y) \cdot \phi(x)\, dy\, dx.$$

By Theorem 6.1.1,

$$\int_\Omega \int_\Omega \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|} \cdot \phi(x) \rho_\delta(x-y)\, dy\, dx =$$
$$\int_\Omega u(x)\, \mathrm{pv}_x \int_\Omega \frac{\phi(x) + \phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y)\, dy\, dx.$$

On the other hand,

$$-\int_{\Omega} u(x)\operatorname{div}^s_\delta \phi(x)\,dx =$$
$$(n+s-1)\int_{\Omega} u(x)\,\mathrm{pv}_x \int_{B(x,\delta)} \frac{\phi(x)+\phi(y)}{|x-y|}\cdot\frac{x-y}{|x-y|}\rho_\delta(x-y)\,dy\,dx,$$

with

$$\mathrm{pv}_x \int_{B(x,\delta)} \frac{\phi(x)+\phi(y)}{|x-y|}\cdot\frac{x-y}{|x-y|}\rho_\delta(x-y)\,dy =$$
$$\mathrm{pv}_x \int_{\Omega} \frac{\phi(x)+\phi(y)}{|x-y|}\cdot\frac{x-y}{|x-y|}\rho_\delta(x-y)\,dy +$$
$$\mathrm{pv}_x \int_{\Omega_{B,\delta}} \frac{\phi(x)+\phi(y)}{|x-y|}\cdot\frac{x-y}{|x-y|}\rho_\delta(x-y)\,dy.$$

Now, for each $x\in\Omega$,

$$\mathrm{pv}_x \int_{\Omega_{B,\delta}} \frac{\phi(x)+\phi(y)}{|x-y|}\cdot\frac{x-y}{|x-y|}\rho_\delta(x-y)\,dy =$$
$$\int_{\Omega_{B,\delta}} \frac{\phi(x)-\phi(y)}{|x-y|}\cdot\frac{x-y}{|x-y|}\rho_\delta(x-y)\,dy = \int_{\Omega_{B,\delta}} \frac{\phi(x)}{|x-y|}\cdot\frac{x-y}{|x-y|}\rho_\delta(x-y)\,dy$$

and these last two integrals are absolutely convergent.

Putting together the formulas above, we obtain the conclusion. □

Note that the minus sign in the boundary term makes sense since the vector $x-y$ points inwards.

We now extend Definition 6.1.1 *a)* to a broader class of functions.

Definition 6.1.2. *Let $0<s<1$, $0<\delta$ and $1\le p<\infty$. Let $u\in L^1(\Omega_\delta)$ be such that there exists a sequence of $\{u_j\}_{j\in\mathbb{N}}\subset C^\infty_c(\mathbb{R}^n)$ converging to u in $L^1(\Omega_\delta)$ and for which $\{D^s_\delta u_j\}_{j\in\mathbb{N}}$ converges to some U in $L^1(\Omega,\mathbb{R}^n)$. We define $D^s_\delta u$ as U.*

The following result shows that the above definitions of $D^s_\delta u$ is independent of the sequence chosen.

Lemma 6.1.3. *Let $0<s<1$ and $0<\delta$. Let $u\in L^1(\Omega_\delta)$ be such that there exist sequences $\{u_j\}_{j\in\mathbb{N}}$ and $\{v_j\}_{j\in\mathbb{N}}$ in $C^\infty_c(\mathbb{R}^n)$ such that $u_j\to u$ and $v_j\to u$ in $L^1(\Omega_\delta)$, and for which $\{D^s_\delta u_j\}_{j\in\mathbb{N}}$ converges to some U and $\{D^s v_j\}_{j\in\mathbb{N}}$ converges to some V in $L^1(\Omega,\mathbb{R}^n)$. Then $U=V$.*

Proof. Let $\phi \in C_c^\infty(\Omega, \mathbb{R}^n)$. By Theorem 6.1.2

$$\int_\Omega U \cdot \phi = \lim_{j\to\infty} \int_\Omega D_\delta^s u_j \cdot \phi =$$
$$- \lim_{j\to\infty} \Bigg(\int_\Omega u_j \operatorname{div}_\delta^s \phi +$$
$$(n+s-1) \int_\Omega \int_{\Omega_{B,\delta}} \frac{u_j(y)\phi(x)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y)\, dy\, dx \Bigg) =$$
$$- \left(\int_\Omega u \operatorname{div}_\delta^s \phi + (n+s-1) \int_\Omega \int_{\Omega_{B,\delta}} \frac{u(y)\phi(x)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y)\, dy\, dx \right)$$

and, analogously,

$$\int_\Omega V \cdot \phi =$$
$$- \left(\int_\Omega u \operatorname{div}_\delta^s \phi + (n+s-1) \int_\Omega \int_{\Omega_{B,\delta}} \frac{u(y)\phi(x)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y)\, dy\, dx \right).$$

Thus,

$$\int_\Omega U \cdot \phi = \int_\Omega V \cdot \phi$$

for all $\phi \in C_c^\infty(\Omega, \mathbb{R}^n)$, whence $U = V$. □

The consideration of the nonlocal gradient as a single object with some analogous properties to the classical one leads to the definition of a new functional space similar to the Sobolev spaces $W^{1,p}(\Omega)$ and the Bessel fractional spaces $H^{s,p}(\mathbb{R}^n)$.

Definition 6.1.3. *Let $1 \leq p < \infty$, $0 < s < 1$ and $0 < \delta$. We define the space $H^{s,p,\delta}(\Omega)$ as*

$$H^{s,p,\delta}(\Omega) := \overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{H^{s,p,\delta}(\Omega)}}$$

equipped with the norm

$$\|u\|_{H^{s,p,\delta}(\Omega)} = \left(\|u\|_{L^p(\Omega_\delta)}^p + \|D_\delta^s u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}.$$

As a consequence of the similarity in its definition, this space enjoys several properties analogous to those of the aforementioned Sobolev spaces.

Proposition 6.1.4. *Set $1 \leq p < \infty$, $0 < s < 1$ and $\delta > 0$. The space $H^{s,p,\delta}(\Omega)$ is a separable Banach space. Moreover, when $p > 1$ it is reflexive.*

Proof. That $H^{s,p,\delta}(\Omega)$ is a Banach space is immediate since its has been defined as a closure

For the rest of the proof, we apply a standard argument; see, for example, [82, Theorem 2.1] for the nonlocal case and [32, Proposition 8.1] for the local case.

We have that the space $F_p = L^p(\Omega_\delta;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^n)$ is separable and, if $p > 1$, it is reflexive. Now we define the map $T : H^{s,p,\delta}(\Omega) \to F_p$ by

$$T(u) = \begin{pmatrix} u \\ D^s_\delta u \end{pmatrix}.$$

Then T is an isometry since

$$\|T(u)\|^p_{F_p} = \|u\|^p_{L^p(\Omega_\delta)} + \|D^s_\delta u\|^p_{L^p(\Omega)} = \|u\|^p_{H^{s,p,\delta}(\Omega)}.$$

By Definitions 6.1.2 and 6.1.3, it is clear that $T(H^{s,p,\delta}(\Omega))$ is a closed subspace of F_p. Since every closed subspace of a reflexive space is reflexive (see, e.g., [32, Proposition 3.20]) and every subset of a separable space is separable (e.g., [32, Proposition 3.25]), it follows that $T(H^{s,p,\delta}(\Omega))$ is separable and, if $p > 1$, it is reflexive. The conclusion follows since T is an isometry. □

In the next result we compare the spaces $H^{s,p,\delta}(\Omega)$ for different exponents p, as well as with the better-known Bessel space $H^{s,p}(\mathbb{R}^n)$.

Proposition 6.1.5. *Let* $1 \leq p < \infty$ *and* $0 < s < 1$.

a) $H^{s,p,\delta}(\Omega) \subset H^{s,q,\delta}(\Omega)$ *whenever* $p \geq q \geq 1$.

b) $H^{s,p}(\mathbb{R}^n) \subset H^{s,p,\delta}(\Omega)$. *In particular, there exists* $C > 0$ *such that for every* $u \in H^{s,p}(\mathbb{R}^n)$,

$$\|u\|_{H^{s,p,\delta}(\Omega)} \leq C\|u\|_{H^{s,p}(\mathbb{R}^n)}.$$

Proof. The proof of *a)* is obtained in a straightforward manner applying the known inclusions $L^p(\Omega_\delta) \subset L^q(\Omega_\delta)$ and $L^p(\Omega) \subset L^q(\Omega)$ to the norms of u and $D^s_\delta u$.

Regarding *b)*, we first prove the corresponding inequality for smooth functions. Thus, let $u \in C_c^\infty(\mathbb{R}^n)$. We have that, for $x \in \Omega$,

$$\begin{aligned} D_\delta^s u(x) =& c_{n,s} \int_{B(x,\delta)} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|^{n+s-1}} \, dy \\ =& c_{n,s} a_0 \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} \, dy \\ &- c_{n,s} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{a_0 - w_\delta(x-y)}{|x-y|^{n+s-1}} \, dy \\ =& a_0 D^s u(x) - c_{n,s} \int_{B(x,b_0\delta)^c} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{a_0 - w_\delta(x-y)}{|x-y|^{n+s-1}} \, dy \\ =& a_0 D^s u(x) + c_{n,s} \int_{B(x,b_0\delta)^c} \frac{u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{a_0}{|x-y|^{n+s-1}} \, dy. \end{aligned}$$

We recall that a_0 and b_0 are the constants from the definition of w_δ and that $w_\delta = a_0$ in $B(0, b_0\delta)$. We therefore have that

$$\begin{aligned} |D_\delta^s u(x)| &\leq a_0 \, |D^s u(x)| + a_0 \, c_{n,s} \int_{B(x,b_0\delta)^c} \frac{|u(y)|}{|x-y|^{n+s}} \, dy \\ &\leq a_0 \, |D^s u(x)| + c_1 \, \|u\|_{L^p(B(x,b_0\delta)^c)} \end{aligned}$$

for some constant $c_1 > 0$. Consequently,

$$\|D_\delta^s u\|_{L^p(\Omega)} \leq a_0 \, \|D^s u\|_{L^p(\Omega)} + c_1 |\Omega|^{\frac{1}{p}} \, \|u\|_{L^p(B(x,b_0\delta)^c))} \leq c_2 \, \|u\|_{H^{s,p}(\mathbb{R}^n)}$$

for some constant $c_2 > 0$. Since we also have that $\|u\|_{L^p(\Omega_\delta)} \leq \|u\|_{L^p(\mathbb{R}^n)}$, we obtain that there exists $C > 0$ such that for every $u \in C_c^\infty(\mathbb{R}^n)$,

$$\|u\|_{H^{s,p,\delta}(\Omega)} \leq C \|u\|_{H^{s,p}(\mathbb{R}^n)}.$$

Being the spaces $H^{s,p,\delta}(\Omega)$ and $H^{s,p}(\mathbb{R}^n)$ defined as the closure of $C_c^\infty(\mathbb{R}^n)$ with their respective norms, the result follows. □

The inclusion from Proposition 6.1.5 *b)* gives a straightforward link with Bessel fractional spaces. In particular, it implies that functions representing fractures and cavitations, as those shown in Section 3.6 are also included in $H^{s,p,\delta}(\Omega)$. One of the advantages of the space $H^{s,p,\delta}(\Omega)$ is that, contrary to $H^{s,p}(\mathbb{R}^n)$, it allows us to deal with elementary functions, such as the identity, which would be relevant in a future linearization process.

Apart from the operator D_δ^s of Definitions 6.1.1 and 6.1.2, there are two closely related operators that are similar to others used in the literature (see [38, 84]).

Definition 6.1.4. *Let $0 < s < 1$ and $\delta > 0$.*

a) Let $u : \Omega_\delta \to \mathbb{R}$ be a measurable function. For $x \in \Omega$, we define

$$\tilde{D}^s_\delta u(x) := c_{n,s}\,\mathrm{pv}_x \int_{B(x,\delta)} \frac{u(x)-u(y)}{|x-y|}\,\frac{x-y}{|x-y|}\,\frac{w_\delta(x-y)}{|x-y|^{n+s-1}}\,dy,$$

whenever the principal value exists.

b) Let $u \in L^p(\Omega_\delta)$. Its distributional nonlocal gradient $\mathcal{D}^s_\delta u$ is defined, for $\phi \in C^\infty_c(\Omega, \mathbb{R}^n)$, as

$$\begin{aligned}\langle \mathcal{D}^s_\delta u, \phi\rangle := &- \int_\Omega u(x)\,\mathrm{div}^s_\delta\,\phi(x)\,dx \\ &- c_{n,s}\int_\Omega \phi(x)\int_{\Omega_{B,\delta}} \frac{u(y)}{|x-y|}\cdot\frac{x-y}{|x-y|}\,\frac{w_\delta(x-y)}{|x-y|^{n+s-1}}\,dy\,dx.\end{aligned}$$

It is immediate to see that these three operators D^s_δ, $\tilde{D}^s_\delta$, $\mathcal{D}^s_\delta$ coincide for smooth functions. In this thesis we deal with D^s_δ and leave for a future work to find out a broader class of functions for which they three coincide.

Finally, notice that, since we considered $u \in C^\infty_c(\Omega)$ in Definition 6.1.1, the function $D^s_\delta u$, originally defined in Ω, also makes sense for points outside Ω, with the same definition. With a small abuse of notation, we also denote by $D^s_\delta u$ the function in $\mathbb{R}^n$ defined as in (6.3).

Before getting to next section we recall the following definitions.

Definition 6.1.5. *We will say that*

a) a function $f : \mathbb{R}^n \to \mathbb{R}$ is radial *if there exists $\bar{f} : \mathbb{R}^+_0 \to \mathbb{R}$ such that $f(x) = \bar{f}(|x|)$ for every $x \in \mathbb{R}^n$.*

b) a radial function $f : \mathbb{R}^n \to \mathbb{R}$ is radially decreasing *if its radial representation $\bar{f} : \mathbb{R}^+_0 \to \mathbb{R}$ is a decreasing function.*

c) a function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is vector radial *if there exists $\bar{\phi} : \mathbb{R}^+_0 \to \mathbb{R}$ such that $\phi(x) = \bar{\phi}(|x|)x$ for every $x \in \mathbb{R}^n$.*

It is known (see, e.g., [66, App. B.5]) that the Fourier transform of a radial (respectively, vector radial) function is radial (respectively, vector radial).

6.2 Calculus in $H^{s,p,\delta}(\Omega)$

We start with a sufficient condition for the nonlocal gradient to be defined everywhere. We recall that a_0 is the constant appearing in the definition of w_δ.

Lemma 6.2.1. *Let $1 \leq p \leq \infty$ and $0 < \delta$. Then there exists a constant $C = C(n, a_0) > 0$ such that for all $u \in W^{1,p}(\Omega_\delta)$. and $0 < s < 1$,*

$$\|D^s_\delta u\|_{L^p(\Omega)} \leq C\delta^{1-s}\frac{c_{n,s}}{1-s}\|Du\|_{L^p(\Omega_\delta)}.$$

Consequently, there exists $C' = C'(n, a_0, |\Omega|) > 0$ such that

$$\|D^s_\delta u\|_{L^r(\Omega)} \leq C'\delta^{1-s}\frac{c_{n,s}}{1-s}\|Du\|_{L^p(\Omega_\delta)}$$

for every $r \in [1, p]$.

Proof. By density, it is enough to prove the equality for $u \in C^\infty_c(\mathbb{R}^n)$.

First we consider $1 \leq p < \infty$, applying Minkowski's integral inequality (see [110, App. A.1]) to the p-norm,

$$\begin{aligned}\left(\int_\Omega \left|\int_{B(x,\delta)} \frac{u(x)-u(y)}{|x-y|}\frac{x-y}{|x-y|}\frac{w_\delta(x-y)}{|x-y|^{n+s-1}}\,dy\right|^p dx\right)^{1/p} \leq \\ \int_{B(0,\delta)}\left(\int_\Omega\left(\frac{|u(x)-u(x-h)|}{|h|^{n+s}}w_\delta(h)\right)^p dx\right)^{1/p} dh.\end{aligned} \tag{6.6}$$

Now, for all $h \in B(0,\delta)\backslash\{0\}$,

$$\begin{aligned}\left(\int_\Omega\left(\frac{|u(x)-u(x-h)|}{|h|^{n+s}}w_\delta(h)\right)^p dx\right)^{1/p} &\leq \frac{a_0}{|h|^{n+s}}\left(\int_\Omega |u(x)-u(x-h)|^p\,dx\right)^{1/p} \\ &\leq \frac{a_0}{|h|^{n+s-1}}\|Du\|_{L^p(\Omega_\delta)}\end{aligned} \tag{6.7}$$

where we have used that w_δ is a bounded function (by $a_0 > 0$) and a classic inequality, [32, Proposition 9.3].

Thus, combining (6.6) and (6.7) we have that

$$\begin{aligned}\|D^s_\delta u\|_{L^p(\Omega)} &\leq c_{n,s}a_0\|Du\|_{L^p(\Omega)}\int_{B(0,\delta)}\frac{1}{|h|^{n+s-1}}\,dh \\ &= \frac{a_0\sigma_{n-1}\delta^{1-s}c_{n,s}}{1-s}\|Du\|_{L^p(\Omega_\delta)}\end{aligned} \tag{6.8}$$

where σ_{n-1} is the area of the unit sphere of $\mathbb{R}^n$. Through a density argument, (6.8) remains true for $u \in W^{1,p}(\Omega_\delta)$.

If $p = \infty$, we apply (6.8) for $q < \infty$. Since for every $f \in L^\infty(\Omega)$, $\|f\|_{L^q(\Omega_\delta)} \le \|f\|_{L^\infty(\Omega_\delta)} |\Omega_\delta|^{\frac{1}{q}}$, we can take a sequence $\{q_m\}_{m\in\mathbb{N}}$ such that $q_m \nearrow \infty$ and

$$\|D^s_\delta u\|_{L^{q_m}(\Omega)} \to \|D^s_\delta u\|_{L^\infty(\Omega)} \qquad \text{and} \qquad \|Du\|_{L^{q_m}(\Omega_\delta)} \to \|Du\|_{L^\infty(\Omega_\delta)}.$$

Applying it in (6.8) we obtain

$$\|D^s_\delta u\|_{L^\infty(\Omega)} \le \frac{a_0 \sigma_{n-1} \delta^{1-s} c_{n,s}}{1-s} \|Du\|_{L^\infty(\Omega_\delta)}.$$

□

Lemma 6.2.1 implies, in particular, that $D^s_\delta \varphi$ is defined everywhere for $\varphi \in C^{0,1}(\Omega_\delta)$. It also shows that $W^{1,p}(\Omega_\delta) \subset H^{s,p,\delta}(\Omega)$ for every $0 < s < 1$ and $1 \le p < \infty$.

Next result is the analogous case in this framework of Lemma 3.3.2.

Lemma 6.2.2. *Let $1 \le q < \infty$, $0 < \delta$ and $0 < s < 1$. Let $\varphi \in C^{0,1}(\Omega_\delta)$ and $k \in \mathbb{N}$. Then, the operator $K^{s,\delta}_\varphi : L^r(\Omega_\delta, \mathbb{R}^{k\times n}) \to L^q(\Omega, \mathbb{R}^k)$ defined as*

$$K^{s,\delta}_\varphi(U)(x) = c_{n,s} \int_{B(x,\delta)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+s}} U(y) \frac{x-y}{|x-y|} w_\delta(x-y) dy, \qquad a.e.\ x \in \Omega,$$

is linear and bounded, for every $r \in [1,q]$, i.e. there exists a constant $C_1 = C_1(n,s,q,\delta,r,w_\delta,|\Omega|) > 0$ such that

$$\|K^{s,\delta}_\varphi(U)\|_{L^r(\Omega,\mathbb{R}^k)} \le [\varphi]^q_{C^{0,1}(\Omega_\delta)} C_1 \|U\|_{L^q(\Omega_\delta,\mathbb{R}^{n\times k})}.$$

$[\varphi]_{C^{0,1}(\Omega_\delta)}$ denotes the Lipschitz semi-norm of φ.

Proof. Let $U \in L^q(\Omega_\delta, \mathbb{R}^{k\times n})$. For a.e. $x \in \Omega$ we have

$$\left| K^{s,\delta}_\varphi(U)(x) \right| \le |c_{n,s}| \int_{B(x,\delta)} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+s}} |U(y)|\, w_\delta(x-y) dy,$$

so

$$\left| K^{s,\delta}_\varphi(U)(x) \right|^q \le |c_{n,s}|^q g(x), \tag{6.9}$$

with

$$g(x) := \left(\int_{B(x,\delta)} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+s}} |U(y)|\, w_\delta(x-y) dy \right)^q,$$

Let $[\varphi]_{C^{0,1}(\Omega_\delta)}$ be the Lipschitz semi-norm of φ. Then, using that φ is Lipschitz and applying Hölder's inequality and the bound $w_\delta(z) \leq C_0$ for every $z \in \mathbb{R}^n$, we get

$$\begin{aligned}
g(x) &\leq [\varphi]^q_{C^{0,1}(\Omega_\delta)} \left(\int_{B(x,\delta)} \frac{|U(y)|}{|x-y|^{n+s-1}} w_\delta(x-y)dy \right)^q \\
&= [\varphi]^q_{C^{0,1}(\Omega_\delta)} C_0^q \left(\int_{B(0,\delta)} \frac{|U(x-z)|}{|z|^{n+s-1}} dz \right)^q \\
&\leq [\varphi]^q_{C^{0,1}(\Omega_\delta)} C_0^q \int_{B(0,\delta)} \frac{|U(x-z)|^q}{|z|^{n+s-1}} dz \left(\int_{B(0,\delta)} \frac{1}{|z|^{n+s-1}} dz \right)^{q-1} \\
&= [\varphi]^q_{C^{0,1}(\Omega_\delta)} C_0^q \left(\frac{\sigma_{n-1}\delta^{1-s}}{1-s} \right)^{q-1} \int_{B(0,\delta)} \frac{|U(x-z)|^q}{|z|^{n+s-1}} dz,
\end{aligned}$$

where σ_{n-1} is the area of the unit sphere of $\mathbb{R}^n$.

Integrating,

$$\begin{aligned}
\int_\Omega g(x)\,dx &\leq [\varphi]^q_{C^{0,1}(\Omega_\delta)} C_0^q \left(\frac{\sigma_{n-1}\delta^{1-s}}{1-s} \right)^{q-1} \int_\Omega \int_{B(0,\delta)} \frac{|U(x-z)|^q}{|z|^{n+s-1}} dz\,dx \\
&= [\varphi]^q_{C^{0,1}(\Omega_\delta)} C_0^q \left(\frac{\sigma_{n-1}\delta^{1-s}}{1-s} \right)^{q-1} \int_{B(0,\delta)} \frac{1}{|z|^{n+s-1}} \int_\Omega |U(x-z)|^q dx\,dz \\
&= [\varphi]^q_{C^{0,1}(\Omega_\delta)} C_0^q \left(\frac{\sigma_{n-1}\delta^{1-s}}{1-s} \right)^{q-1} \|U\|^q_{L^q(\Omega_\delta,\mathbb{R}^{k\times n})} \int_{B(0,\delta)} \frac{1}{|z|^{n+s-1}} dz \\
&\leq [\varphi]^q_{C^{0,1}(\Omega_\delta)} C_0^q \left(\frac{\sigma_{n-1}\delta^{1-s}}{1-s} \right)^{q} \|U\|^q_{L^q(\Omega_\delta,\mathbb{R}^{k\times n})} .
\end{aligned} \tag{6.10}$$

Thus, by (6.9) we have

$$\|K^{s,\delta}_\varphi(U)\|_{L^q(\Omega,\mathbb{R}^k)} \leq [\varphi]^q_{C^{0,1}(\Omega_\delta)} |c_{n,s}|^q C_0^q \left(\frac{\sigma_{n-1}\delta^{1-s}}{1-s} \right)^q \|U\|_{L^q(\Omega_\delta,\mathbb{R}^{n\times k})}.$$

Therefore, for every $r \in [1,q]$ we have that there exists a constant $C_1 = C_1(n,s,q,\delta,r,w_\delta,|\Omega|) > 0$ such that

$$\|K^{s,\delta}_\varphi(U)\|_{L^r(\Omega,\mathbb{R}^k)} \leq [\varphi]^q_{C^{0,1}(\Omega_\delta)} C_1 \|U\|_{L^q(\Omega_\delta,\mathbb{R}^{n\times k})}.$$

And the proof is completed. $\square$

As a consequence of Lemma 6.2.2 and a general result, the operator $K_{\varphi}^{s,\delta}$ is continuous from the weak topology of $L^q(\Omega_\delta, \mathbb{R}^{k\times n})$ to the weak topology of $L^p(\Omega, \mathbb{R}^k)$ for all $p \in [1, q]$.

Now we introduce a product formula for the nonlocal gradient. We denote by I the identity matrix of dimension n. Notice that these computations would also be admissible with operator $\tilde{D}_\delta^s u$ from Definition 6.1.4.

Lemma 6.2.3. *Let $0 < s < 1$, $0 < \delta$ and $1 < p < \infty$. Let $g \in H^{s,p,\delta}(\Omega)$ and $\varphi \in C^1(\overline{\Omega}_\delta)$. Then $\varphi g \in H^{s,p,\delta}(\Omega)$ and for a.e. $x \in \Omega$,*

$$D_\delta^s(\varphi g)(x) = \varphi(x) D_\delta^s g(x) + K_\varphi^{s,\delta}(gI)(x).$$

Proof. Clearly $\varphi g \in L^p(\mathbb{R}^n)$. First we consider $g \in C_c^\infty(\mathbb{R}^n)$ and later we will extend it by density. Now, for a.e. $x \in \Omega$ we have

$$\frac{1}{c_{n,s}} D_\delta^s(\varphi g)(x) = \mathrm{pv}_x \int_{B(x,\delta)} \frac{(\varphi g)(x) - (\varphi g)(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} w_\delta(x-y)\, dy =$$

$$\mathrm{pv}_x \int_{B(x,\delta)} \frac{\varphi(x)g(x) - \varphi(x)g(y) + \varphi(x)g(y) - \varphi(y)g(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} w_\delta(x-y)\, dy =$$

$$\frac{1}{c_{n,s}} \left(\varphi(x) D_\delta^s g(x) + K_\varphi^{s,\delta}(gI)(x) \right).$$

The term $\varphi\, D_\delta^s g$ is in $L^p(\Omega, \mathbb{R}^n)$ since $\varphi \in C_c^1(\mathbb{R}^n)$, while the term $K_\varphi^{s,\delta}(gI)$ is in $L^p(\Omega, \mathbb{R}^n)$ by Lemma 6.2.2 .

Now we consider $g \in H^{s,p.\delta}(\Omega)$ and a sequence $\{g_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $\{g_j\}_{j\in\mathbb{N}}$ converges to g in $L^p(\Omega_\delta)$ and $\{D_\delta^s g_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega, \mathbb{R}^n)$. It is immediate to check that $\varphi\, u_j \to \varphi\, u$ in $L^p(\Omega_\delta)$. Let us check that $\{D_\delta^s(\varphi\, u_j)\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega, \mathbb{R}^n)$. Owing to Lemma 6.2.3 we have

$$D_\delta^s(\varphi\, u_j) - D_\delta^s(\varphi\, u_k) = D_\delta^s(\varphi(u_j - u_k)) = \varphi\, D_\delta^s(u_j - u_k) + K_\varphi^{s,\delta}(u_j - u_k),$$

for $j, k \in \mathbb{N}$. Next, since $D_\delta^s(u_j - u_k) \to 0$ in $L^p(\Omega, \mathbb{R}^n)$ as $j, k \to \infty$, we also have that $\varphi\, D_\delta^s(u_j - u_k) \to 0$ in $L^p(\Omega, \mathbb{R}^n)$. By Lemma 6.2.2 , since $u_j - u_k \to 0$ in $L^p(\Omega_\delta)$ as $j, k \to \infty$, we obtain that $K_\varphi^{s,\delta}(u_j - u_k) \to 0$ in $L^p(\Omega, \mathbb{R}^n)$. This shows that $\varphi\, u \in H^{s,p,\delta}(\Omega)$. ☐

Its proof is analogous to that of [84, Lemma 2.3], and, hence, it will be omitted.

For a $\phi \in H^{s,p,\delta}(\Omega, \mathbb{R}^n)$ there is a natural relation between $D_\delta^s \phi$ and $\mathrm{div}_\delta^s\, \phi$.

Lemma 6.2.4. *Let $0 < s < 1$ and $1 \leq p < \infty$. Let $\phi \in H^{s,p,\delta}(\Omega, \mathbb{R}^n)$. Then $\mathrm{div}_\delta^s\, \phi$ is well defined and $\mathrm{tr}\, D_\delta^s \phi = \mathrm{div}_\delta^s\, \phi$ a.e.*

Proof. By density, it suffices to show that $\operatorname{tr} D^s_\delta \phi = \operatorname{div}^s_\delta \phi$ for all $\phi \in C^\infty_c(\mathbb{R}^n)$. Having in mind that the integrals of (6.5) and of the right hand side of (6.4) are absolutely convergent, we obtain that

$$\begin{aligned}
\operatorname{tr} D^s_\delta \phi(x) &= c_{n,s} \operatorname{tr}\left(\int_{B(x,\delta)} \frac{\phi(x)-\phi(y)}{|x-y|^{n+s}} \otimes \frac{x-y}{|x-y|} w_\delta(x-y)dy\right) \\
&= c_{n,s} \int_{B(x,\delta)} \operatorname{tr}\left(\frac{\phi(x)-\phi(y)}{|x-y|^{n+s}} \otimes \frac{x-y}{|x-y|} w_\delta(x-y)\right) dy \\
&= c_{n,s} \int_{B(x,\delta)} \frac{\phi(x)-\phi(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} w_\delta(x-y)dy = \operatorname{div}^s_\delta \phi(x),
\end{aligned}$$

which concludes the proof. □

As in Lemma 6.2.3, the following result computes the nonlocal divergence of a product.

Lemma 6.2.5. *Let $0 < s < 1$, $0 < \delta$ and $1 < p < \infty$. Let $g \in H^{s,p,\delta}(\Omega, \mathbb{R}^n)$ and $\varphi \in C^1_c(\Omega)$. Then $\varphi g \in H^{s,p,\delta}(\Omega, \mathbb{R}^n)$ and for a.e. $x \in \mathbb{R}^n$,*

$$\operatorname{div}^s_\delta(\varphi g)(x) = \varphi(x) \operatorname{div}^s_\delta g(x) + K^{s,\delta}_\varphi(g^T)(x).$$

Finally we show an analogous of the Schwartz theorem of the symmetry of second derivatives.

Proposition 6.2.6. *Let $0 < s < 1$, $0 < \delta$ and $u \in C^\infty_c(\mathbb{R}^n)$. Let $i \in \{1, \dots, n\}$ and*

$$D^s_{\delta,i} u(x) := (n+s-1) \int_{B(x,\delta)} \frac{u(x)-u(y)}{|x-y|} \frac{x_i - y_i}{|x-y|} \rho_\delta(x-y)\, dy. \tag{6.11}$$

Then, for every $i, j \in \{1, \dots, n\}$, the following equality holds

$$D^s_{\delta,j}(D^s_{\delta,i} u) = D^s_{\delta,i}(D^s_{\delta,j} u).$$

Proof. We have that, making the change of variables $y = \bar{y} + x' - x$ and renaming $\bar{y} = y$,

$$\begin{aligned}
&\frac{D^s_{\delta,i} u(x) - D^s_{\delta,i} u(x')}{(n+s-1)} = \\
&\int_{B(x,\delta)} \frac{u(x)-u(y)}{|x-y|} \frac{x_i - y_i}{|x-y|} \rho_\delta(x-y)\, dy - \\
&\int_{B(x',\delta)} \frac{u(x')-u(y)}{|x'-y|} \frac{x'_i - y_i}{|x'-y|} \rho_\delta(x'-y)\, dy = \\
&\int_{B(x,\delta)} \frac{[u(x)-u(y)-u(x')+u(y+x'-x)]}{|x-y|^2} (x_i - y_i) \rho_\delta(x-y)\, dy.
\end{aligned}$$

Therefore,

$$\frac{D^s_{\delta,j}(D^s_{\delta,i}u)(x)}{(n+s-1)^2} = \int_{B(x,\delta)}\int_{B(x,\delta)} \frac{[u(x)-u(y)-u(x')+u(y+x'-x)]}{|x-y|^2|x-x'|^2} (x_i-y_i)\rho_\delta(x-y)(x_j-x'_j)\rho_\delta(x-x')\,dydx'. \tag{6.12}$$

If we apply know Fubini theorem we obtain that

$$\frac{D^s_{\delta,j}(D^s_{\delta,i}u)(x)}{(n+s-1)^2} = \int_{B(x,\delta)} \frac{\int_{B(x,\delta)}\frac{u(x)-u(x')}{|x-x'|^2}(x_j-x'_j)\rho_\delta(x-x')dx'}{|x-y|^2}(x_i-y_i)\rho_\delta(x-y)dy- \int_{B(x,\delta)} \frac{\int_{B(x,\delta)}\frac{u(y)-u(y+x'-x)}{|x-x'|^2}(x_j-x'_j)\rho_\delta(x-x')dx'}{|x-y|^2}(x_i-y_i)\rho_\delta(x-y)dy.$$

If we make the change of variables $\bar{x} = x' - x + y$, it yields

$$\begin{aligned}\int_{B(x,\delta)} \frac{u(y)-u(y+x'-x)}{|x-x'|^2}(x_j-x'_j)\rho_\delta(x-x')dx = \\ \int_{B(y,\delta)} \frac{u(y)-u(\bar{x})}{|y-\bar{x}|^2}(y_j-\bar{x}_j)\rho_\delta(y-\bar{x})d\bar{x} = \frac{D^s_{\delta,j}u(y)}{(n+s-1)}.\end{aligned} \tag{6.13}$$

Therefore, combining (6.12) and (6.13), we have that

$$\frac{D^s_{\delta,j}(D^s_{\delta,i}u)(x)}{(n+s-1)^2} = \int_{B(x,\delta)} \frac{D^s_{\delta,j}u(x)-D^s_{\delta,j}u(y)}{|x-y|^2}(x_i-y_i)\rho_\delta(x-y)\,dy \frac{D^s_{\delta,i}(D^s_{\delta,j}u)(x)}{(n+s-1)^2},$$

and the results follows. □

Remark 6.2.1. *If we define each component of the nonlocal divergence of a field $\phi \in C^\infty_c(\mathbb{R}^n,\mathbb{R}^n)$ as*

$$\operatorname{div}^s_{\delta,i}\phi_i(x) = -\operatorname{pv}_x(n+s-1)\int_{B(x,\delta)} \frac{\phi_i(x)+\phi_i(y)}{|x-y|}\frac{x_i-y_i}{|x-y|}\rho_\delta(x-y)\,dy, \tag{6.14}$$

by odd symmetry we have that

$$\operatorname{div}^s_{\delta,i} \phi_i(x) = D^s_{\delta,i}\phi_i(x). \tag{6.15}$$

Then, and as a consequence of Proposition 6.2.6, we have that for $u \in C^\infty_c(\mathbb{R}^n)$

$$\operatorname{div}^s_{\delta,j}(D^s_{\delta,i}u) = \operatorname{div}^s_{\delta,i}(D^s_{\delta,j}u) \quad \text{and} \quad \operatorname{div}^s_{\delta,j}(D^s_{\delta,i}u) = D^s_{\delta,j}(\operatorname{div}^s_{\delta,i} u).$$

6.3 Nonlocal version of the Fundamental Theorem of Calculus

We start by recalling the following classical representation theorem which can be seen in [63, Lemma 7.14] or [92, Prop. 4.14]. Although it has already been introduced in Section 3.5, given the analysis performed in this section, we show it again here for the sake of the reader.

Proposition 6.3.1. *For every* $\varphi \in C^\infty_c(\mathbb{R}^n)$ *and every* $x \in \mathbb{R}^n$*, we have*

$$\varphi(x) = \frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} \nabla\varphi(y) \cdot \frac{x-y}{|x-y|^n}\, dy,$$

where σ_{n-1} *is the area of the unit sphere.*

This result may be understood as a fundamental theorem of calculus, in the sense that we recover a function from its gradient by integration. A fractional version of it, involving the Riesz fractional gradient in the whole space is also known (see Theorem 3.5.1). This section is devoted to show a novel nonlocal version of Proposition 6.3.1, where a function can be recovered from its nonlocal gradient D^s_δ through a convolution with a suitable kernel V^s_δ.

Our approach is inspired by the proofs of the fractional fundamental theorem of calculus previously referred in [92, 99]. However, those rely on the semigroup properties of Riesz potentials, which our kernels do not enjoy. Therefore our procedure is much more involved.

To begin with, we show that the kernel in the definition of $D^s_\delta u$ (see formula (6.3)) can be seen, in a certain sense, as the gradient of a certain function.

Lemma 6.3.2. *Let $0 < s < 1$ and $0 < \delta$. Define $q_\delta : [0,\infty) \to \mathbb{R}$ and $Q^s_\delta : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ as*

$$q_\delta(t) = (n+s-1)t^{n+s-1} \int_t^\delta \frac{\bar{w}_\delta(r)}{r^{n+s}}\, dr \qquad \text{and}$$
$$Q^s_\delta(\tilde{x}) = \frac{1}{\gamma(1-s)|\tilde{x}|^{n+s-1}} q_\delta(|\tilde{x}|).$$

Then:

a) q_δ is $C^\infty((0,\infty))$, $C^{n-1}([0,\infty))$, $q_\delta|_{[\delta,\infty)} = 0$ and there exists a constant $z_0 \in \mathbb{R}$ such that for every $t \in [0, b_0\delta]$,

$$q_\delta(t) = a_0 + z_0\, t^{n+s-1}.$$

b) Q^s_δ is radially decreasing, $Q^s_\delta \in L^1(\mathbb{R}^n)$ and

$$\frac{-1}{n+s-1} \nabla Q^s_\delta(\tilde{x}) = \frac{\rho_\delta(\tilde{x})}{|\tilde{x}|} \frac{\tilde{x}}{|\tilde{x}|}. \tag{6.16}$$

Proof. We start with *a)*. The function q_δ is clearly C^∞ in $(0,\infty)$ as a product of C^∞ functions in $(0,\infty)$. We have that

$$\left(\frac{q_\delta(t)}{t^{n+s-1}}\right)' = -(n+s-1)\frac{\bar{w}_\delta(t)}{t^{n+s}}, \qquad t > 0. \tag{6.17}$$

Since $q_\delta(\delta) = 0$ and $\bar{w}_\delta|_{[\delta,\infty)} = 0$, we obtain that $q_\delta|_{[\delta,\infty)} = 0$. Now, for $0 < t < \delta b_0$ we have that

$$\begin{aligned} q_\delta(t) &= (n+s-1)t^{n+s-1} \left(\int_t^{b_0\delta} \frac{\bar{w}_\delta(r)}{r^{n+s}}\, dr + \int_{b_0\delta}^{\delta} \frac{\bar{w}_\delta(r)}{r^{n+s}}\, dr \right) \\ &= a_0 \left(1 - \left(\frac{t}{b_0\delta} \right)^{n+s-1} \right) + (n+s-1)t^{n+s-1} \int_{b_0\delta}^{\delta} \frac{\bar{w}_\delta(r)}{r^{n+s}}\, dr, \end{aligned}$$

where a_0 and b_0 are the constants from the definition of w_δ. In particular, q_δ is $C^{n-1}([0,\infty))$ and the existence of z_0 in the statement follows.

We now show *b)*. We get immediately from (6.17) that

$$\nabla Q^s_\delta(\tilde{x}) = -\frac{n+s-1}{\gamma(1-s)} \frac{\bar{w}_\delta(|\tilde{x}|)}{|\tilde{x}|^{n+s}} \frac{\tilde{x}}{|\tilde{x}|}, \qquad \tilde{x} \in \mathbb{R}^n \setminus \{0\},$$

so (6.16) holds. The fact that Q^s_δ is radially decreasing function is also obtained from (6.17). Finally, $Q^s_\delta \in L^1(\mathbb{R}^n)$ as a consequence of the boundedness of q_δ. $\square$

Next, we show the following proposition, whose goal is to write the nonlocal gradient as a convolution of the classical one with a kernel. Its proof, inspired by that of [92, Lemma 15.9] (see also [99], which is stated in this document in Theorem 3.1.6), is based on an integration by parts starting with (6.16). Recall, from the comments after Definition 6.1.4, that $D^s_\delta u$ is defined in the whole $\mathbb{R}^n$ for $u \in C^\infty_c(\mathbb{R}^n)$. In this case, $\operatorname{supp} D^s_\delta u \subset \operatorname{supp} u + B(0,\delta)$.

Proposition 6.3.3. *For every $u \in C^\infty_c(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we have*

$$D^s_\delta u(x) = \int_{\mathbb{R}^n} \nabla u(y) Q^s_\delta(x-y) dy \tag{6.18}$$

and $D^s_\delta u \in C^\infty_c(\mathbb{R}^n)$.

Proof. Let K be a ball containing $\operatorname{supp}\varphi$ and let $K_\delta = K + B(0,\delta)$. If $x \in K^c_\delta$ then both terms of (6.18) are zero since $\operatorname{supp} D^s_\delta \varphi \subset K_\delta$ and $\operatorname{supp} Q^s_\delta \subset B(0,\delta)$. Thus, we consider $x \in K_\delta$, $e \in \mathbb{R}^n$ with $|e| = 1$ and the vector field

$$\beta : K_\delta \setminus \{x\} \to \mathbb{R}^n$$

defined by

$$\beta(y) = (u(x) - u(y)) Q^s_\delta(x-y)e.$$

Let $\varepsilon > 0$ be such that $\bar{B}(x,\varepsilon) \subset K_\delta$. From Lemma 6.3.2 we have that

$$\operatorname{div}\beta(y) = (n+s-1)\frac{u(x)-u(y)}{|x-y|}\rho_\delta(x-y)\frac{x-y}{|x-y|} \cdot e - Q^s_\delta(x-y)\nabla u(y) \cdot e, \tag{6.19}$$

for $y \in K_\delta \setminus B(x,\varepsilon)$. Notice also that $\operatorname{div}\beta$ is integrable in $K_\delta \setminus B(x,\varepsilon)$. Applying the divergence theorem we obtain

$$\int_{K_\delta \setminus B(x,\varepsilon)} \operatorname{div}\beta(y)\, dy = \int_{\partial K_\delta} \beta(y)\cdot\nu_y \, d\mathcal{H}^{n-1}(y) + \int_{\partial B(x,\varepsilon)} \beta(y)\cdot\frac{x-y}{|x-y|}\, d\mathcal{H}^{n-1}(y),$$

where ν_y is the outer normal vector to K_δ. Now we show that $\beta(y) = 0$ for all $y \in \partial K_\delta$. Indeed, if $x \in K_\delta \setminus K$ then $u(x) = u(y) = 0$ for all $y \in \partial K_\delta$, whereas if $x \in K$, then $Q^s_\delta(x-y) = 0$ for every $y \in \partial K_\delta$. Thus,

$$\int_{K_\delta \setminus B(x,\varepsilon)} \operatorname{div}\beta(y)\, dy = \int_{\partial B(x,\varepsilon)} \beta(y) \cdot \frac{x-y}{|x-y|}\, d\mathcal{H}^{n-1}(y).$$

Now we estimate the integrand in the right-hand side. As u is Lipschitz, using the definition of Q^s_δ (see Lemma 6.3.2) we find that, for all $y \in \partial B(x,\varepsilon)$,

$$\begin{aligned}\left|\beta(y) \cdot \frac{x-y}{|x-y|}\right| &\le |\beta(y)| \le \|\nabla u\|_\infty |x-y|\, |Q^s_\delta(x-y)| \le \|\nabla u\|_\infty \frac{c}{|x-y|^{n+s-2}} \\ &= \|\nabla u\|_\infty \frac{c}{\varepsilon^{n+s-2}}\end{aligned}$$

for some constant $c > 0$, so

$$\left| \int_{\partial B(x,\varepsilon)} \beta(y) \cdot \frac{x-y}{|x-y|} \, d\mathcal{H}^{n-1}(y) \right| \leq \|\nabla u\|_\infty \, c\sigma_{n-1}\varepsilon^{1-s},$$

which goes to 0 when ε goes to 0. Therefore,

$$\lim_{\varepsilon \to 0} \int_{K_\delta \setminus B(x,\varepsilon)} \operatorname{div} \beta(y) dy = 0.$$

As a result, using (6.19) we obtain that

$$\lim_{\varepsilon \to 0} \int_{K_\delta \setminus B(x,\varepsilon)} (n+s-1) \frac{u(x)-u(y)}{|x-y|} \rho_\delta(x-y) \frac{x-y}{|x-y|} \cdot e \, dy =$$
$$\lim_{\varepsilon \to 0} \int_{K_\delta \setminus B(x,\varepsilon)} Q_\delta^s(x-y) \nabla u(y) \cdot e \, dy,$$

provided that both limits exists, which is actually true as both integrals are absolutely convergent in K_δ; see the comment after Definition 6.1.4 for the left integral and notice that $Q_\delta^s \in L^1(\mathbb{R}^n)$ (see Lemma 6.3.2) for the right integral. Thus,

$$\int_{K_\delta} (n+s-1) \frac{u(x)-u(y)}{|x-y|} \rho_\delta(x-y) \frac{x-y}{|x-y|} \cdot e \, dy = \int_{K_\delta} Q_\delta^s(x-y) \nabla u(y) \cdot e \, dy.$$

As this is true for every $e \in \mathbb{R}^n$ with $|e| = 1$, we conclude that

$$\int_{K_\delta} (n+s-1) \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \rho_\delta(x-y) \, dy = \int_{K_\delta} \nabla u(y) Q_\delta^s(x-y) \, dy,$$

and formula (6.18) is proved.

We have thus shown that $D_\delta^s u = \nabla u * Q_\delta^s$. As $Q_\delta^s \in L^1(\mathbb{R}^n)$ and both u and Q_δ^s have compact support, this implies that $D_\delta^s u \in C_c^\infty(\mathbb{R}^n)$. □

Given the previous result, it is straightforward to obtain the following corollary (for more details on the Fourier transform, see next subsection).

Corollary 6.3.4. *Let $0 < s < 1$ and $0 < \delta$. Then, for all $u \in C_c^\infty(\mathbb{R}^n)$,*

$$\widehat{D_\delta^s u}(\xi) = 2\pi i \xi \, \hat{u}(\xi) \, \hat{Q}_\delta^s(\xi).$$

Proof. Given $u \in C_c^\infty(\mathbb{R}^n)$ we apply Fourier transform on (6.18) (a convolution of two functions in $L^1(\mathbb{R}^n)$). Thus we obtain

$$\widehat{D_\delta^s u}(\xi) = \widehat{\nabla u}(\xi) \hat{Q}_\delta^s(\xi) = 2\pi i \xi \hat{u}(\xi) \hat{Q}_\delta^s(\xi).$$

□

We continue with the two main results of this section.

Proposition 6.3.5. *Let $0 < s < 1$ and $0 < \delta$. Then there exists a unique function $V_\delta^s \in C^\infty(\mathbb{R}^n \backslash \{0\})$ such that*

$$\int_{\mathbb{R}^n} V_\delta^s(z) Q_\delta^s(y-z) dz = \frac{1}{\sigma_{n-1}} \frac{y}{|y|^n}, \qquad y \in \mathbb{R}^n \setminus \{0\}. \tag{6.20}$$

Moreover, $V_\delta^s \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$, and for every $R > 0$ there exists $M > 0$ such that

$$|V_\delta^s(x)| \leq \frac{M}{|x|^{n-s}}, \qquad x \in B(0,R) \setminus \{0\}.$$

For further properties of V_δ^s see Theorem 6.3.14.

The proof of Proposition 6.3.5 is long and comprises the whole of Section 6.3.1. With this, the main theorem of this section reads as follows. Its proof follows the lines from [92, Prop. 15.8], whereas the main differences are gathered in Proposition 6.3.5.

Theorem 6.3.6. *Let $0 < s < 1$ and $0 < \delta$. Let V_δ^s be the function of Proposition 6.3.5. Then, for every $u \in C_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$u(x) = \int_{\mathbb{R}^n} D_\delta^s u(y) \cdot V_\delta^s(x-y)\, dy. \tag{6.21}$$

Proof. Let $F(x)$ denote the right hand side of (6.21). This integral is absolutely convergent since $V_\delta^s \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ (Proposition 6.3.5) and $D_\delta^s u$ is bounded with compact support (Proposition 6.3.3). In fact, Proposition 6.3.3 allows us to write the equality

$$F(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \nabla u(z) Q_\delta^s(y-z) \cdot V_\delta^s(x-y)\, dz\, dy.$$

Next we make the changes of variables $\eta = x - y$ and $\xi = x - z$ to obtain

$$F(x) = \int_{\mathbb{R}^n} \nabla u(x-\xi) \cdot \int_{\mathbb{R}^n} V_\delta^s(\eta) Q_\delta^s(\xi - \eta)\, d\eta\, d\xi.$$

By Proposition 6.3.5,

$$\int_{\mathbb{R}^n} V_\delta^s(\eta) Q_\delta^s(\xi-\eta) d\eta = \frac{1}{\sigma_{n-1}} \frac{\xi}{|\xi|^n}.$$

Thus, thanks to Proposition 6.3.1,

$$F(x) = \int_{\mathbb{R}^n} \nabla u(x-\xi) \cdot \frac{\xi}{\sigma_{n-1}|\xi|^n} d\xi = u(x)$$

and the proof is complete. □

6.3.1 Inverse Kernel

This section is devoted to the proof of Proposition 6.3.5, which is divided into several intermediate results. In the first half of the section, we will see that formula

$$\hat{V}_\delta^s(\xi) = \frac{-i\xi}{2\pi|\xi|^2} \frac{1}{\hat{Q}_\delta^s(\xi)}$$

(obtained heuristically by applying Fourier transform to (6.20) and using Lemma 6.3.19) is well defined. The crucial point for this is to show that $\hat{Q}_\delta^s$ is positive. We then conclude that V_δ^s is at least a tempered distribution. In the second half, we see that V_δ^s is actually a function.

We indicate the convention for the Fourier transform of a function that we use:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx$$

for $f \in L^1(\mathbb{R}^n)$. This definition is extended by continuity and duality to many other function and distribution spaces, notably, as isomorphisms in L^2, in the Schwartz space $\mathcal{S}$ and in the space of tempered distributions $\mathcal{S}'$. Sometimes we will also use the alternative notation $\mathcal{F}(f)$ for $\hat{f}$. Classical texts in Fourier analysis are [66, 71].

In a great part of the proof, we will make comparisons with the Riesz potential. Although it has been extensively used in Part II, we recall (see [99, 110]) that given $0 < s < n$, the Riesz potential $I_s : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ and, in particular, its Fourier transform are

$$I_s(x) = \frac{1}{\gamma(s)} \frac{1}{|x|^{n-s}} \quad \text{and} \quad \hat{I}_s(\xi) = |2\pi\xi|^{-s}, \tag{6.22}$$

where $\gamma(s)$ is defined in (6.2).

We start with an analysis of the Fourier transform of the vectorial version of the q_δ of Lemma 6.3.2.

Lemma 6.3.7. *Let $0 < s < 1$ and $0 < \delta$. Let $\mathbf{q}_\delta : \mathbb{R}^n \to \mathbb{R}$ be defined as $\mathbf{q}_\delta(x) = q_\delta(|x|)$. Then its Fourier transform is an analytic function and $\hat{\mathbf{q}}_\delta \in C_0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.*

Proof. Half of this proof comes directly from known facts in Fourier analysis. Given that $q_\delta \in C_c^{n-1}([0,\infty))$, we have that $\mathbf{q}_\delta \in L^1(\mathbb{R}^n)$ and therefore $\hat{\mathbf{q}}_\delta \in C_0(\mathbb{R}^n)$. Actually, since $\mathbf{q}_\delta$ has compact support, $\hat{\mathbf{q}}_\delta$ is analytic. Now we check that $\mathbf{q}_\delta \in W^{2n-1,1}(\mathbb{R}^n)$. Indeed, as a consequence of Lemma 6.3.2, for $1 \leq j \leq 2n-1$ there exists $z_j \in \mathbb{R}$ such that

$$q_\delta^{j)}(t) = z_j t^{n+s-1-j}, \qquad t \in (0, b_0\delta),$$

where the superindex j indicates the j-th derivative. On the other hand, $q_\delta^{j)}$ is bounded in $[b_0\delta, \delta]$ and vanishes in $[\delta, \infty)$. This implies that the a.e. and weak derivative of order j of $\mathbf{q}_\delta$ coincide and they satisfy, for some constant $C_j > 0$,

$$\left|D^{j)}(x)\right| \leq C_j \, |x|^{n+s-1-j}, \qquad x \in B(0, b_0\delta) \setminus \{0\}$$

while $\left|D^{j)}(x)\right|$ is bounded in $B(0,\delta) \setminus B(0, b_0, \delta)$ and vanishes in $B(0,\delta)^c$. This implies that $\mathbf{q}_\delta \in W^{2n-1,1}(\mathbb{R}^n)$. In particular, the Fourier transform of any partial derivative of order $2n-1$ of $\mathbf{q}_\delta$ is bounded, so there exists $C > 0$ such that for any multiindex α of order $2n-1$ we have

$$|(2\pi i\xi)^\alpha \hat{\mathbf{q}}_\delta(\xi)| = |\widehat{\partial^\alpha \mathbf{q}_\delta}(\xi)| \leq C,$$

and, hence,

$$|\hat{\mathbf{q}}_\delta(\xi)| \leq \frac{C}{|2\pi\xi|^{2n-1}}.$$

This decay at infinity of $\hat{\mathbf{q}}_\delta$, together with the fact that $\hat{\mathbf{q}}_\delta$ is continuous, implies that $\hat{\mathbf{q}}_\delta \in L^1(\mathbb{R}^n)$ for $n \geq 2$.

In the rest of the proof, we assume that $n = 1$. In this case, $\mathbf{q}_\delta$ is the even extension of q_δ. As shown before, there exists $z_1 \in \mathbb{R}$ such that $\mathbf{q}_\delta'(x) = \frac{z_1}{|x|^{1-s}}$ for $x \in B(0, b_0\delta)$.

Now we consider a $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi|_{B(0,\frac{1}{4})} = 1$ and $\varphi|_{B(0,\frac{1}{2})^c} = 0$. Then,

$$\begin{aligned}|2\pi\xi|^{-s} - \frac{1}{z_1\gamma(s)}\widehat{\mathbf{q}'}_\delta(\xi) &= \mathcal{F}\left(\frac{1}{\gamma(s)|x|^{1-s}} - \frac{1}{z_1\gamma(s)}\mathbf{q}_\delta'(x)\right) \\ &= \mathcal{F}\left(\frac{\varphi}{\gamma(s)|x|^{1-s}} - \frac{1}{z_1\gamma(s)}\mathbf{q}_\delta'(x)\right) + \mathcal{F}\left(\frac{1-\varphi}{\gamma(s)|x|^{1-s}}\right).\end{aligned}$$

Looking at the expression of $\mathbf{q}_\delta'$, we notice that the functions $\frac{\varphi}{\gamma(s)|x|^{1-s}}$ and $\frac{1}{z_1\gamma(s)}\mathbf{q}_\delta'(x)$ coincide in $B(0, \min\{b_0\delta, \frac{1}{4}\})$, and both have compact support. Therefore, its difference is a smooth function of compact support. It particular it is in the Schwartz space, as well as its Fourier transform:

$$\mathcal{F}\left(\frac{\varphi}{\gamma(s)|x|^{1-s}} - \frac{1}{z_1\gamma(s)}\mathbf{q}_\delta'(x)\right) \in \mathcal{S}.$$

On the other hand, the function $\mathcal{F}\left(\frac{1-\varphi}{\gamma(s)|x|^{1-s}}\right)$ is treated in [66, Ex. 2.4.9], and it is concluded that its decay at infinity is faster than any negative power of $|\xi|$. Consequently, the decay at infinity of

$$|2\pi\xi|^{-s} - \frac{1}{z_1\gamma(s)}\widehat{\mathbf{q}'}_\delta(\xi)$$

is also faster than any negative power of $|\xi|$. In particular, there exists $C_1' > 0$ such that

$$\left|\frac{2\pi\xi}{z_1\gamma(s)}\hat{\mathbf{q}}_\delta(\xi)\right| < \frac{C_1'}{|2\pi\xi|^s},$$

which allows us to conclude that $\hat{\mathbf{q}}_\delta \in L^1(\mathbb{R})$. $\square$

The last lemma is useful so as to obtain relevant properties about $\hat{Q}_\delta^s$.

Proposition 6.3.8. *Let $0 < s < 1$ and $0 < \delta$. Then*

a) $\hat{Q}_\delta^s$ is analytic, bounded, radial, and $\hat{Q}_\delta^s(0) = \|Q_\delta^s\|_{L^1(\mathbb{R}^n)}$.

b) For every multi-index α, $\partial^\alpha \hat{Q}_\delta^s$ is bounded.

c) $\displaystyle\lim_{|\xi|\to\infty} \frac{\hat{Q}_\delta^s(\xi)}{|2\pi\xi|^{-(1-s)}} = a_0,$

where a_0 is the constant from the definition of w_δ.

Proof. The proof of part *a)* comes directly from known facts in Fourier analysis. Indeed, as $Q_\delta^s \in L^1(\mathbb{R}^n)$ we have $\hat{Q}_\delta^s \in L^\infty(\mathbb{R}^n)$. As Q_δ^s has compact support, $\hat{Q}_\delta^s$ is analytic. Since Q_δ^s is radial, so is $\hat{Q}_\delta^s$. Finally, the equality $\hat{Q}_\delta^s(0) = \|Q_\delta^s\|_{L^1(\mathbb{R}^n)}$ is a straightforward consequence of the formula of the Fourier transform.

Regarding *b)*, we have that $\partial^\alpha \hat{Q}_\delta^s = \mathcal{F}\left((-2\pi i\xi)^\alpha Q_\delta^s\right)$. Thus, $\partial^\alpha \hat{Q}_\delta^s$ is the Fourier transform of an $L^1(\mathbb{R}^n)$ function (since $Q_\delta^s \in L^1(\mathbb{R}^n)$ has compact support). Therefore, $\partial^\alpha \hat{Q}_\delta^s$ is bounded.

In order to show *c)*, we apply the Fourier transform to the expression $Q_\delta^s = I_{1-s}\,\mathbf{q}_\delta$ (see the definition in Lemmas 6.3.2 and 6.3.7). Since the Riesz potential I_{1-s} is not an $L^1(\mathbb{R}^n)$ function and $\mathbf{q}_\delta$ is not Schwartz, the Fourier transform is, in principle, in the sense of tempered distributions. To wit, as $I_{1-s} \in L^1(B(0,1)) + L^\infty(B(0,1)^c)$, both factors I_{1-s} and $\mathbf{q}_\delta$ can be seen as distributions; in addition, $\mathbf{q}_\delta$ has compact support, so we can use Lemma 6.3.20 and obtain that

$$\hat{Q}_\delta^s(\xi) = |2\pi\xi|^{-(1-s)} * \hat{\mathbf{q}}_\delta(\xi) \tag{6.23}$$

in the sense of distributions. Actually, by Young's inequality for the convolution we have that

$$\|\hat{I}_{1-s} * \hat{\mathbf{q}}_\delta\|_\infty \le \|\hat{I}_{1-s}\chi_{B(0,1)}\|_1 \|\hat{\mathbf{q}}_\delta\|_\infty + \|\hat{I}_{1-s}\chi_{B(0,1)^c}\|_\infty \|\hat{\mathbf{q}}_\delta\|_1.$$

Therefore, the integral defining $(\hat{I}_{1-s} * \hat{\mathbf{q}}_\delta)(\xi)$ is absolutely convergent for a.e. $\xi \in \mathbb{R}^n$. Consequently, equality (6.23) holds a.e., and, since $\hat{Q}^s_\delta$ is continuous, it holds everywhere.

Then, we consider $\xi = \lambda\xi_0$ with $\xi_0 \in B(0, 1/2)^c$ fixed and $\lambda > 0$. Using the change of variables $x = \lambda x'$ we have

$$\begin{aligned}\hat{Q}^s_\delta(\lambda\xi_0) &= \int |2\pi(x - \lambda\xi_0)|^{-(1-s)} \hat{\mathbf{q}}_\delta(x)\, dx \\ &= \int |2\pi(\lambda x - \lambda\xi_0)|^{-(1-s)} \hat{\mathbf{q}}_\delta(\lambda x)\lambda^n\, dx \\ &= \lambda^{-(1-s)} \int |2\pi(\xi_0 - x)|^{-(1-s)} \hat{\mathbf{q}}_\delta(\lambda x)\lambda^n\, dx.\end{aligned}$$

As the function $\xi \mapsto \frac{\hat{Q}^s_\delta(\xi)}{|2\pi\xi|^{-(1-s)}}$ is radial, in order for *c)* to hold, it is enough that

$$\lim_{\lambda\to\infty} \frac{\hat{Q}^s_\delta(\lambda\xi_0)}{|2\pi\lambda\xi_0|^{-(1-s)}} = a_0,$$

equivalently,

$$\lim_{\lambda\to\infty} \frac{\int |2\pi(\xi_0 - x)|^{-(1-s)} \hat{\mathbf{q}}_\delta(\lambda x)\lambda^n\, dx}{|2\pi\xi_0|^{-(1-s)}} = a_0.$$

Define now $g_\lambda(x) = \frac{1}{a_0}\hat{\mathbf{q}}_\delta(\lambda x)\lambda^n$. The limit above is equivalent to

$$\lim_{\lambda\to\infty} \frac{\int |2\pi(\xi_0 - x)|^{-(1-s)} g_\lambda(x)\, dx}{|2\pi\xi_0|^{-(1-s)}} = 1.$$

Define now $f(\xi) = |2\pi\xi|^{-(1-s)}$. The limit above is equivalent to

$$\lim_{\lambda\to\infty} \frac{\int f(\xi_0 - x) g_\lambda(x)\, dx}{f(\xi_0)} = 1,$$

and, in turn, equivalent to

$$\lim_{\lambda\to\infty} \int f(\xi_0 - x) g_\lambda(x)\, dx = f(\xi_0),$$

in other words,

$$\lim_{\lambda\to\infty} f * g_\lambda(\xi_0) = f(\xi_0).$$

We recall from Lemmas 6.3.2 and 6.3.7 that $a_0 = \mathbf{q}_\delta(0) = \int \hat{\mathbf{q}}_\delta$. Thus, $\int g_\lambda = 1$ for each $\lambda > 0$. Then, by construction, g_λ is a mollifier family tending to the Dirac delta at 0, when $\lambda \to \infty$ in the sense of distributions. Thus,

$$\begin{aligned}|f * g_\lambda(\xi_0) - f(\xi_0)| &= \left| \int_{\mathbb{R}^n} f(\xi_0 - x) g_\lambda(x)\, dx - \int_{\mathbb{R}^n} f(\xi_0) g_\lambda(x)\, dx \right| \\ &\leq \int_{\mathbb{R}^n} |f(\xi_0 - x) - f(\xi_0)|\, |g_\lambda(x)|\, dx.\end{aligned}$$

Let $\varepsilon > 0$. Since f is uniformly continuous in $B(0, 1/2)^c$, there exists $r > 0$ such that

$$|f(\xi_0 - x) - f(\xi_0)| < \varepsilon \quad \forall \xi_0 \in B(0, 1/2)^c, \forall x \in B(0, r).$$

Therefore, as $f \in L^\infty(B(0, 1/2)^c)$, in fact,

$$\|f\|_{L^\infty(B(0,1/2)^c)} = f(1/2) = \pi^{-(1-s)} \leq 1,$$

we have that

$$\begin{aligned}|f * g_\lambda(\xi_0) - f(\xi_0)| \leq &\int_{B(0,r)} |f(\xi_0 - x) - f(\xi_0)|\, |g_\lambda(x)|\, dx \\ &+ \int_{B(0,r)^c} |f(\xi_0 - x) - f(\xi_0)|\, |g_\lambda(x)|\, dx \\ \leq &\varepsilon \int_{B(0,r)} |g_\lambda(x)| dx + 2\|f\|_{L^\infty(B(0,1/2)^c)} \int_{B(0,r)^c} |g_\lambda(x)| dx.\end{aligned}$$

Finally, we use that g_λ is a mollifier sequence, in particular, we have that $\lim_{\lambda \to \infty} \int_{B(0,r)^c} |g_\lambda(x)|\, dx \to 0$. As a result, there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, the inequality $\int_{B(0,r)^c} |g_\lambda(x)|\, dx < \varepsilon$ holds. As a consequence

$$|f * g_\lambda(\xi_0) - f(\xi_0)| \leq (\|g_\lambda\|_{L^1(\mathbb{R}^n)} + 2\|f\|_{L^\infty(B(0,1/2)^c)})\varepsilon \leq (\|g_1\|_{L^1(\mathbb{R}^n)} + 2)\varepsilon,$$

since $\|g_\lambda\|_1 = \|g_1\|_1$. This proves the convergence $f * g_\lambda(\xi_0) \to f(\xi_0)$ for every $\xi_0 \in B(0, 1/2)^c$, when $\lambda \to \infty$, and, hence, the convergence of the statement. □

The next step is the positivity of $\hat{Q}^s_\delta$. For such a process, the following calculation is useful.

Lemma 6.3.9. *Let $0 < s < 1$ and $0 < \delta$. Then, for all $j \in \{1, \dots, n\}$ and $r > 0$,*

$$\int \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi r x_j)\, dx > 0.$$

Proof. By a change of variables we have

$$\int \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi r x_j)\, dx = r^s \int \frac{x_j}{|x|^{n+s+1}} w_\delta\left(\frac{x}{r}\right) \sin(2\pi x_j)\, dx.$$

Let $\bar{w}_\delta$ be the radial representation of w_δ. By symmetry, the co-area formula and Fubini's Theorem we have

$$\begin{aligned}
&\int \frac{x_j}{|x|^{n+s+1}} w_\delta\left(\frac{x}{r}\right) \sin(2\pi x_j)\, dx = \\
&2\int_{\{x_j>0\}} \frac{x_j}{|x|^{n+s+1}} w_\delta\left(\frac{x}{r}\right) \sin(2\pi x_j)\, dx = \\
&2\int_0^{r\delta} \frac{\bar{w}_\delta\left(\frac{t}{r}\right)}{t^{n+s+1}} t^{n-1} \int_{\mathbb{S}_j^+} t z_j \sin(2\pi t z_j)\, d\mathcal{H}^{n-1}(z)\, dt = \\
&2\int_{\mathbb{S}_j^+} z_j \int_0^{r\delta} \frac{\bar{w}_\delta\left(\frac{t}{r}\right)}{t^{s+1}} \sin(2\pi t z_j)\, dt\, d\mathcal{H}^{n-1}(z),
\end{aligned}$$

where $\mathbb{S}_j^+ = \{z \in \mathbb{S}^{n-1} : z_j > 0\}$. Finally, let us show that

$$\int_0^{r\delta} \frac{\bar{w}_\delta\left(\frac{t}{r}\right)}{t^{s+1}} \sin(2\pi t z_j)\, dt > 0$$

for each $z \in \mathbb{S}_j^+$ and $r > 0$. For this, consider the function $f(t) = \frac{\bar{w}_\delta\left(\frac{t}{r}\right)}{t^{s+1}}$ and express

$$\int_0^{r\delta} \frac{\bar{w}_\delta\left(\frac{t}{r}\right)}{t^{s+1}} \sin(2\pi t z_j)\, dt = \sum_{k=0}^{\infty} \int_{\frac{k}{z_j}}^{\frac{k+1}{z_j}} f(t) \sin(2\pi t z_j)\, dt.$$

We have that each term in the sum is positive; indeed, by splitting the integral in two through points $\frac{k+\frac{1}{2}}{z_j}$ and making the change of variables $t = t' + \frac{1}{2z_j}$ in one of them, it is easy to obtain

$$\int_{\frac{k}{z_j}}^{\frac{k+1}{z_j}} f(t) \sin(2\pi t z_j)\, dt = \int_{\frac{k}{z_j}}^{\frac{k+\frac{1}{2}}{z_j}} \left[f(t) - f(t + \frac{1}{2z_j})\right] \sin(2\pi t z_j)\, dt \geq 0,$$

since $\sin(2\pi t z_j) > 0$ and f is decreasing. In fact,

$$\int_0^{\frac{1}{2z_j}} \left[f(t) - f(t + \frac{1}{2z_j})\right] \sin(2\pi t z_j)\, dt > 0,$$

as f is strictly decreasing in $[0,\alpha]$ for some $\alpha > 0$, so

$$\sum_{k=0}^{\infty} \int_{\frac{k}{z_j}}^{\frac{k+1}{z_j}} f(t)\sin(2\pi t z_j)\,dt > 0,$$

which concludes the proof. □

We conclude the first half of the subsection with the following result.

Proposition 6.3.10. *Let $0 < s < 1$ and $0 < \delta$. Then $\hat{Q}^s_\delta(\xi) > 0$ for all $\xi \in \mathbb{R}^n$.*

Proof. Since $\hat{Q}^s_\delta(0) = \|Q^s_\delta\|_{L^1(\mathbb{R}^n)} > 0$, we have to show that $\hat{Q}^s_\delta(\xi) > 0$ for every $\xi \in \mathbb{R}^n \setminus \{0\}$. In order to do so, taking into account that $\hat{Q}^s_\delta$ is radial, we will see that

$$2\pi i \xi_j \hat{Q}^s_\delta(\xi_j e_j) = \widehat{\frac{\partial Q^s_\delta}{\partial x_j}}(\xi_j e_j) \neq 0 \qquad \xi_j > 0,$$

with e_j the j-th vector of the canonical basis. Now, we claim that, despite $\frac{\partial Q^s_\delta}{\partial x_j} \notin L^1(\mathbb{R}^n)$,

$$\widehat{\frac{\partial Q^s_\delta}{\partial x_j}}(\xi) = \frac{(n+s-1)}{\gamma(1-s)} i \int \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi \xi \cdot x)\,dx. \tag{6.24}$$

This is shown at the end of the proof. Assuming the validity of (6.24), by Lemma 6.3.9 we obtain

$$\frac{1}{i}\widehat{\frac{\partial Q^s_\delta}{\partial x_j}}(\xi_j e_j) > 0, \qquad \xi_j > 0.$$

Now, the formula

$$2\pi i \xi_j \hat{Q}^s_\delta(\xi_j e_j) = \widehat{\frac{\partial Q^s_\delta}{\partial x_j}}(\xi_j e_j)$$

holds in the sense of tempered distributions. Since both terms are actually functions, the equality holds as functions for almost every point. Moreover, since both functions are continuous, the equality holds everywhere. We then conclude that

$$\xi_j \hat{Q}^s_\delta(\xi_j e_j) > 0, \qquad \xi_j > 0.$$

Consequently, since $\hat{Q}^s_\delta$ is radial, $\hat{Q}^s_\delta(\xi) > 0$ for all $\xi \in \mathbb{R}^n$.

It remains to prove (6.24). By Lemma 6.3.2,

$$\frac{\partial Q_\delta^s}{\partial x_j}(x) = -(n+s-1)\frac{\rho_\delta(x)}{|x|}\frac{x_j}{|x|}.$$

We do have $\frac{\partial Q_\delta^s}{\partial x_j}\chi_{B(0,\varepsilon)^c} \in L^1(\mathbb{R}^n)$ for all $\varepsilon > 0$. Moreover, by Lemma 6.3.15

$$\frac{\partial Q_\delta^s}{\partial x_j}\chi_{B(0,\varepsilon)^c} \to \frac{\partial Q_\delta^s}{\partial x_j} \quad \text{in } \mathcal{S}' \quad \text{as } \varepsilon \to 0,$$

so

$$\mathcal{F}\left(\frac{\partial Q_\delta^s}{\partial x_j}\chi_{B(0,\varepsilon)^c}\right) \to \mathcal{F}\left(\frac{\partial Q_\delta^s}{\partial x_j}\right) \quad \text{in } \mathcal{S}' \quad \text{as } \varepsilon \to 0.$$

We now compute

$$\begin{aligned}\mathcal{F}\left(\frac{\partial Q_\delta^s}{\partial x_j}\chi_{B(0,\varepsilon)^c}\right)(\xi) &= -\frac{(n+s-1)}{\gamma(1-s)}\int_{B(0,\varepsilon)^c}\frac{x_j}{|x|^{n+s+1}}w_\delta(x)e^{-2\pi i\xi\cdot x}\,dx\\ &= \frac{(n+s-1)}{\gamma(1-s)}i\int_{B(0,\varepsilon)^c}\frac{x_j}{|x|^{n+s+1}}w_\delta(x)\sin\left(2\pi\xi\cdot x\right)dx,\end{aligned}$$

where we have used the odd symmetry. Now, by dominated convergence

$$\int_{B(0,\varepsilon)^c}\frac{x_j}{|x|^{n+s+1}}w_\delta(x)\sin\left(2\pi\xi\cdot x\right)dx \to \int\frac{x_j}{|x|^{n+s+1}}w_\delta(x)\sin\left(2\pi\xi\cdot x\right)dx$$

since

$$\begin{aligned}\left|\chi_{B(0,\varepsilon)^c}\frac{x_j}{|x|^{n+s+1}}w_\delta(x)\sin\left(2\pi\xi\cdot x\right)\right| &\le \frac{1}{|x|^{n+s}}w_\delta(x)\left|2\pi\xi\cdot x\right|\\ &\le \frac{1}{|x|^{n+s-1}}w_\delta(x)2\pi\left|\xi\right|\end{aligned}$$

and

$$\int\frac{1}{|x|^{n+s-1}}w_\delta(x)\,dx < \infty.$$

This proves (6.24). □

Corollary 6.3.11. *Let $0 < s < 1$, $0 < \delta$. There exists a tempered distribution V_δ^s whose Fourier transform is given by*

$$\hat{V}_\delta^s(\xi) = -\frac{i\xi}{2\pi|\xi|^2}\frac{1}{\hat{Q}_\delta^s(\xi)}. \tag{6.25}$$

Proof. The right-hand side of (6.25) is well defined for $\xi \in \mathbb{R}^n \setminus \{0\}$ since $\hat{Q}^s_\delta$ is positive (Proposition 6.3.10). Next, we subtract the function (and distribution) $\frac{-\xi}{|\xi|}\frac{1}{|2\pi\xi|}$ of Lemma 6.3.19 from $\hat{V}^s_\delta$:

$$\hat{V}^s_\delta(\xi) - \frac{-i\xi}{2\pi|\xi|^2}\frac{1}{\hat{Q}^s_\delta(0)} = -\frac{i\xi}{2\pi|\xi|^2}\frac{1}{\hat{Q}^s_\delta(\xi)} - \frac{-i\xi}{2\pi|\xi|^2}\frac{1}{\hat{Q}^s_\delta(0)}. \tag{6.26}$$

This function is in $L^\infty(B(0,R)^c)$ for any $R > 0$, as a difference of functions in $L^\infty(B(0,R)^c)$ (see Proposition 6.3.8). Let us see that it is also in $L^\infty(B(0,R))$ for some $R > 0$. By a Taylor expansion, there exist $c > 0$ and $R > 0$ such that for all $\xi \in B(0,R)$,

$$\left|\hat{Q}^s_\delta(0) - \hat{Q}^s_\delta(\xi)\right| \leq c|\xi|.$$

As a result,

$$\left|\hat{V}^s_\delta(\xi) - \frac{-i\xi}{2\pi|\xi|^2}\frac{1}{\hat{Q}^s_\delta(0)}\right| = \frac{1}{2\pi|\xi|}\frac{|\hat{Q}^s_\delta(0) - \hat{Q}^s_\delta(\xi)|}{\hat{Q}^s_\delta(\xi)\hat{Q}^s_\delta(0)} \leq \frac{1}{2\pi}\frac{c}{\hat{Q}^s_\delta(0)\min_{B(0,R)}\hat{Q}^s_\delta},$$

so the function in (6.26) is in $L^\infty(B(0,R))$, and, hence, in $L^\infty(\mathbb{R}^n)$. In particular, this function is a tempered distribution, and so is $\hat{V}^s_\delta$. As the Fourier transform is an isomorphism from $\mathcal{S}'$ into itself, there exists $V^s_\delta \in \mathcal{S}'$ such that (6.25) holds. □

In this second part we address the existence of V^s_δ as a function. First we notice that $\hat{V}^s_\delta$ does not belong to any space where we can conclude directly that its Fourier transform is a function. The main drawback comes from the fact that the tail of $\hat{V}^s_\delta$ is not integrable enough, although in the limit it behaves like a homogeneous function with a known Fourier transform. So as to tackle this, we adapt the proof of [66, Proposition 2.4.8] (homogeneous function) to the non-homogeneous function $\hat{V}^s_\delta$.

We first need the following decay estimate for the derivatives of $\hat{V}^s_\delta$. We use the multiindex notation for the higher-order partial derivatives.

Lemma 6.3.12. *For every $\alpha \in \mathbb{N}^n$ there exists $C_\alpha > 0$ such that for any $|\xi| \geq 1$,*

$$\left|\partial^\alpha \hat{V}^s_\delta(\xi)\right| \leq \frac{1}{|\xi|^{s(|\alpha|+1)}}.$$

Proof. Express $\hat{V}^s_\delta = g\,f$ with

$$g(\xi) = \frac{\xi}{|\xi|}, \qquad f = f_1 \circ g_1, \qquad f_1(t) = t^{-1}, \qquad g_1(\xi) = |\xi|\hat{Q}^s_\delta(\xi).$$

By Leibniz' formula,

$$\partial^\alpha (gf) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta g \partial^{\alpha-\beta} f.$$

Let $\beta \in \mathbb{N}^n$ and $j \in \mathbb{N}$. Denote by g^j the j-th component of g. By induction, it is easy to see that $\partial^\beta g^j(\xi)$ can be expressed as

$$\frac{P(\xi)}{|\xi|^{2|\beta|+1}}$$

for some polynomial P of degree $|\beta|+1$. Therefore,

$$\left|\partial^\beta g(\xi)\right| \leq \frac{C_\beta}{|\xi|^{|\beta|}}, \qquad \xi \in \mathbb{R}^n \setminus \{0\}. \tag{6.27}$$

We apply Faà di Bruno's formula for the higher-order derivatives of a composition, and obtain that

$$\partial^\gamma f = \sum_{i=1}^{|\gamma|} f_1^{i)} \circ g_1 \, G_i$$

where G_i is a linear combination of products of i partial derivatives of g_1, the order of which adds up $|\gamma|$.

We estimate the partial derivatives of g_1. We express $g_1 = h\,\hat{Q}^s_\delta$ with $h(\xi) = |\xi|$. Since $\nabla h = g$, we have, by (6.27), that

$$\left|\partial^\beta h(\xi)\right| \leq \frac{C_\beta}{|\xi|^{|\beta|-1}}, \qquad \xi \in \mathbb{R}^n \setminus \{0\}. \tag{6.28}$$

Now we show that

$$|\partial^\beta \hat{Q}^s_\delta(\xi)| \leq \frac{C_\beta}{|\xi|^{|\beta|}}, \qquad \xi \in \mathbb{R}^n \setminus \{0\}. \tag{6.29}$$

Recalling from Lemma 6.3.2 the definition of Q^s_δ, we mention here that it is an L^1 function of compact support, smooth outside the origin, and that in a ball B centred at the origin, one has

$$Q^s_\delta(x) = A_0 + \frac{C_0}{|x|^{n+s-1}}, \qquad x \in B \setminus \{0\}$$

for some $A_0, C_0 \in \mathbb{R}$. With this expression it is easy to see that

$$\left|\partial^\beta (x^\beta Q^s_\delta(x))\right| = \left|\sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \partial^\gamma \left(x^\beta\right) \partial^{\gamma-\beta} Q^s_\delta(x)\right| \leq \frac{C_\beta}{|x|^{n+s-1}}, \qquad x \in B \setminus \{0\}$$

for some $C_\beta \in \mathbb{R}$. Moreover, $\partial^\beta((-2\pi x)^\beta Q_\delta^s)$ is smooth outside the origin, has compact support and is in L^1. Consequently, $\mathcal{F}\left(\partial^\beta((-2\pi x)^\beta Q_\delta^s)\right)$ is analytic and bounded. But

$$\mathcal{F}\left(\partial^\beta((-2\pi x)^\beta Q_\delta^s)\right) = (2\pi\xi)^\beta \mathcal{F}((-2\pi x)^\beta Q_\delta^s) = (2\pi\xi)^\beta \partial^\beta \hat{Q}_\delta^s(\xi),$$

which shows (6.29).

Now, by Leibniz' formula, (6.28) and (6.29),

$$|\partial^\alpha g_1(\xi)| \leq C_\alpha \sum_{\beta\leq\alpha} \left|\partial^\beta h(\xi)\right| \left|\partial^{\alpha-\beta}\hat{Q}_\delta^s(\xi)\right| \leq \frac{C_\alpha}{|\xi|^{|\alpha|-1}}, \qquad \xi \in \mathbb{R}^n \setminus \{0\},$$

for some constant $C_\alpha > 0$. Hence, if we multiply i partial derivatives of g_1, the order of which adds up $|\gamma|$, we obtain that

$$|G_i(\xi)| \leq \frac{C_{\gamma,i}}{|\xi|^{|\gamma|-i}}, \qquad \xi \in \mathbb{R}^n \setminus \{0\},$$

for some constants $C_{\gamma,i} > 0$. On the other hand, by induction,

$$\left|f_1^{i)}(t)\right| \leq \frac{C_i}{t^{i+1}}, \qquad i \in \mathbb{N}, \quad t > 0,$$

for some constants $C_i > 0$, and, hence,

$$\left|f_1^{i)} \circ g_1(\xi)\right| \leq \frac{C_i}{\left(|\xi|\hat{Q}_\delta^s(\xi)\right)^{i+1}}, \qquad i \in \mathbb{N}, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

From Proposition 6.3.8 we know that, for $|\xi| \geq 1$,

$$\left|\frac{1}{\hat{Q}_\delta^s(\xi)}\right| \leq C\,|\xi|^{1-s},$$

so

$$\frac{1}{\left(|\xi|\hat{Q}_\delta^s(\xi)\right)^{i+1}} \leq \frac{C}{|\xi|^{s(i+1)}}.$$

Thus,

$$|\partial^\gamma f| \leq \sum_{i=1}^{|\gamma|} \left|f_1^{i)} \circ g_1\right| |G_i| \leq C_\gamma \sum_{i=1}^{|\gamma|} \frac{1}{|\xi|^{s(i+1)+|\gamma|-i}} \leq \frac{C_\gamma}{|\xi|^{s(|\gamma|+1)}}.$$

We conclude that, for $|\xi| \geq 1$,

$$\left|\partial^\alpha \hat{V}_\delta^s\right| \leq C_\alpha \sum_{\beta\leq\alpha} \left|\partial^\beta g\right| \left|\partial^{\alpha-\beta} f\right| \leq C_\alpha \sum_{\beta\leq\alpha} \frac{1}{|\xi|^{s(|\alpha|+1)+|\beta|(1-s)}} \leq \frac{C_\alpha}{|\xi|^{s(|\alpha|+1)}}.$$

□

The decay estimate of Lemma 6.3.12 is not optimal. In fact, a more refined argument would possibly allow us to prove that the bound

$$|\partial^\alpha \hat{Q}^s_\delta(\xi)| \leq \frac{C_\alpha}{|\xi|^{|\beta|+1-s}}, \qquad \xi \in \mathbb{R}^n$$

holds. With that estimate, an adaptation of the proof of Lemma 6.3.12 would yield

$$\left|\partial^\alpha \hat{V}^s_\delta(x)\right| \leq \frac{C_\alpha}{1+|x|^{|\alpha|+s}}.$$

Nevertheless, the bound of Lemma 6.3.12 is enough for our purposes in the following theorem. Before that, we need the following inverse Lipschitz estimate of the function $\frac{x}{|x|^{n-s+1}}$.

Lemma 6.3.13. *Let $0 < s < 1$. For every $R_1, R_2 > 0$ there exists $m > 0$ such that, for all $x \in B(0,R_1) \setminus \{0\}$ and $h \in B(0,R_2) \setminus \{x\}$,*

$$m|h| \leq \left| \frac{x}{|x|^{n+1-s}} - \frac{x-h}{|x-h|^{n+1-s}} \right|. \tag{6.30}$$

Proof. We divide the proof into four cases, according to the position of the points x and h. Let us define $G(x) = \frac{x}{|x|^{n+1-s}}$.

Case 1: $2|x| \leq |x-h|$. Taking

$$m \leq \frac{1 - \frac{1}{2^{n-s}}}{R_1^{n-s} R_2}$$

we have

$$\begin{aligned} |G(x) - G(x-h)| &\geq \frac{1}{|x|^{n-s}} - \frac{1}{|x-h|^{n-s}} \geq \left(1 - \frac{1}{2^{n-s}}\right) \frac{1}{|x|^{n-s}} \\ &\geq \frac{1 - \frac{1}{2^{n-s}}}{R_1^{n-s}} \geq m R_2 \geq m|h|. \end{aligned}$$

Case 2: $G(x) \cdot G(x-h) \leq 0$. Taking

$$m \leq \frac{1}{R_1^{n-s} R_2}$$

we have

$$\begin{aligned} &|G(x) - G(x-h)| = \Big(|G(x)|^2 + |G(x-h)|^2 - 2G(x)\cdot G(x-h)\Big)^{\frac{1}{2}} \geq |G(x)| \\ &= \frac{1}{|x|^{n-s}} \geq \frac{1}{R_1^{n-s}} \geq mR_2 \geq m|h|. \end{aligned}$$

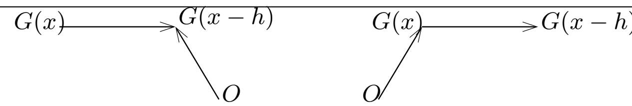


Figure 6.1: Position of the points $G(x)$, $G(x-h)$ and the origin O when $(G(x-h)-G(x))\cdot G(x-h) \leq 0$ (left) and when $(G(x-h)-G(x))\cdot G(x) \geq 0$ (right)

Case 3: $|x-h| \leq 2|x|$ and

$$\min\left\{|G(x)|^2, |G(x-h)|^2\right\} \leq G(x) \cdot G(x-h). \tag{6.31}$$

We observe that the inverse of G is $G^{-1}(y) = \frac{y}{|y|^{\frac{n+1-s}{n-s}}}$, with derivative

$$DG^{-1}(y) = |y|^{-\frac{n-s+1}{n-s}-2} y \otimes y + |y|^{-\frac{n-s+1}{n-s}} I,$$

so

$$|DG^{-1}(y)| \leq 2|y|^{-\frac{n-s+1}{n-s}}. \tag{6.32}$$

With this, using the mean value theorem,

$$\begin{aligned} |h| &= |G^{-1}(G(x)) - G^{-1}(G(x-h))| \\ &\leq \left\|DG^{-1}\right\|_{L^\infty([G(x),G(x-h)])} |G(x) - G(x-h)|. \end{aligned} \tag{6.33}$$

Now,

$$\begin{aligned} \|DG^{-1}\|_{L^\infty([G(x),G(x-h)])} &\leq 2 \max_{y\in[G(x),G(x-h)]} |y|^{-\frac{n-s+1}{n-s}} \\ &= 2\left(\min_{y\in[G(x),G(x-h)]} |y|\right)^{-\frac{n-s+1}{n-s}}. \end{aligned}$$

Elementary geometry shows that

$$\min_{y\in[G(x),G(x-h)]} |y| = \begin{cases} |G(x-h)| & \text{if } (G(x-h)-G(x)) \cdot G(x-h) \leq 0, \\ |G(x)| & \text{if } (G(x-h)-G(x)) \cdot G(x) \geq 0; \end{cases} \tag{6.34}$$

see Figure 6.1. Assumption (6.31) asserts that one of these two options occurs, so

$$\min_{y\in[G(x),G(x-h)]} |y| \geq \min\{|G(x-h)|, |G(x)|\}$$

and, hence,

$$\left(\min_{y\in[G(x),G(x-h)]}|y|\right)^{-\frac{n-s+1}{n-s}} \leq \left(\min\left\{|G(x-h)|,|G(x)|\right\}\right)^{-\frac{n-s+1}{n-s}} = \max\{|x|^{n-s+1},|x-h|^{n-s+1}\}.$$

Finally, since $|x-h|\leq 2|x|$,

$$\max\{|x|^{n-s+1},|x-h|^{n-s+1}\} \leq 2^{n-s+1}|x|^{n-s+1} \leq 2^{n-s+1}R_1^{n-s+1}. \tag{6.35}$$

Going back to (6.33), we find that $|h|\leq 2^{n-s+2}R_1^{n-s+1}|G(x)-G(x-h)|$, so inequality (6.30) holds for

$$m \leq \frac{1}{2^{n-s+2}R_1^{n-s+1}}.$$

Case 4: $|x-h|\leq 2|x|$ and

$$0 < G(x)\cdot G(x-h) < \min\left\{|G(x)|^2,|G(x-h)|^2\right\}. \tag{6.36}$$

Note first that inequality (6.36) cannot occur in dimension $n=1$.

Let $\gamma:[0,1]\to\mathbb{R}^n$ be any piecewise C^1 curve such that $\gamma(0)=G(x)$ and $\gamma(1)=G(x-h)$. By the fundamental theorem of Calculus,

$$|h| = \left|G^{-1}(\gamma(0))-G^{-1}(\gamma(1))\right| = \left|\int_0^1 \left(G^{-1}\circ\gamma\right)'(t)\,dt\right| \leq \max_{\gamma([0,1])}\left|DG^{-1}\right|\ell(\gamma), \tag{6.37}$$

where ℓ denotes the length of the curve.

Assumption (6.36) implies that none of the cases of (6.34) occurs (hence none of the situations depicted in Figure 6.1), but the distance from the origin to the segment $[G(x),G(x-h)]$ is attained at a point P in the interior of the segment. Assume that $|G(x)-P|\leq|G(x-h)-P|$, although the construction is totally analogous in the symmetric case $|G(x)-P|\geq|G(x-h)-P|$. Let Q be the point in the segment $[G(x),G(x-h)]$ such that P is the middle point between $G(x)$ and Q. We define the curve γ as follows. The curve γ starts at $G(x)$ and describes the arc of circumference of center the origin O and radius $|G(x)|$ joining $G(x)$ with Q; among the two possible arcs, we choose that which subtends an angle of less than π radians. Then, γ continues joining Q and $G(x-h)$ with a straight line. See Figure 6.2. For this particular γ we estimate the right hand-side of (6.37). First, using (6.32),

$$\max_{\gamma([0,1])}\left|DG^{-1}\right| \leq 2\max_{y\in\gamma([0,1])}|y|^{-\frac{n-s+1}{n-s}} = 2|G(x)|^{-\frac{n-s+1}{n-s}}, \tag{6.38}$$

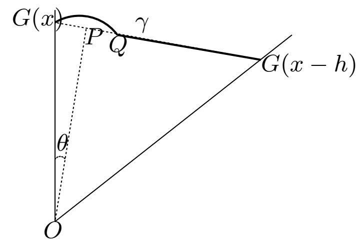


Figure 6.2: The curve γ (in thick line), the points $G(x), P, Q, G(x-h)$ (aligned, in dotted line), the origin O and the angle θ

since, by construction of γ, the shortest distance of $\gamma([0,1])$ to the origin is $|G(x)|$. In order to estimate $\ell(\gamma)$, let θ be the angle $\widehat{G(x)\,O\,P}$ if it is positive, or else the opposite angle $\widehat{P\,O\,G(x)}$, so that

$$\sin\theta = \frac{\ell([G(x),P])}{|G(x)|}$$

and $\theta \in [0, \frac{\pi}{2}]$ because $0 \le G(x)\cdot G(x-h)$. Then

$$\ell(\gamma) = 2\theta\,|G(x)| + \ell([Q, G(x-h)]).$$

Now we use the elementary inequality

$$t \le \frac{\pi}{2}\sin t, \qquad t \in [0, \frac{\pi}{2}]$$

to obtain that

$$2\theta\,|G(x)| \le \pi\sin\theta\,|G(x)| = \pi\,\ell([G(x),P]) = \frac{\pi}{2}\,\ell([G(x),Q]),$$

so

$$\ell(\gamma) \le \frac{\pi}{2}\,\ell([G(x),Q]) + \ell([Q,G(x-h)]) \le \frac{\pi}{2}\,\ell([G(x),G(x-h)]). \tag{6.39}$$

Using (6.38) and (6.39), inequality (6.37) becomes

$$|h| \le \pi\,|G(x)|^{-\frac{n-s+1}{n-s}}\;\ell([G(x),G(x-h)]).$$

If we had assumed $|G(x)-P| \ge |G(x-h)-P|$ instead of $|G(x)-P| \le |G(x-h)-P|$ we would have obtained

$$|h| \le \pi\,|G(x-h)|^{-\frac{n-s+1}{n-s}}\;\ell([G(x),G(x-h)]),$$

so, in either case,

$$\begin{aligned}|h| &\leq \pi \max\left\{|G(x)|^{-\frac{n-s+1}{n-s}}, |G(x-h)|^{-\frac{n-s+1}{n-s}}\right\} \ell([G(x), G(x-h)]) \\ &= \pi \max\left\{|x|^{n-s+1}, |x-h|^{n-s+1}\right\} |G(x) - G(x-h)|\,.\end{aligned}$$

Now we use (6.35) and find that inequality (6.30) holds for

$$m \leq \frac{1}{2^{n-s+2}\pi R_1^{n-s+1}}.$$

□

Theorem 6.3.14. *Let $0 < s < 1$ and $0 < \delta$. Then there exists a radial function $V_\delta^s \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$ such that*

$$\hat{V}_\delta^s(\xi) = -i\frac{\xi}{|\xi|}\frac{1}{|2\pi\xi|\hat{Q}_\delta^s(\xi)}. \tag{6.40}$$

Furthermore, we have the following properties:

a) There exists $W \in C_b(\mathbb{R}^n, \mathbb{R}^n)$ (actually, $W \in C_0(\mathbb{R}^n, \mathbb{R}^n)$ when $n \geq 2$) such that

$$V_\delta^s(x) = W(x) + \frac{c_{n,-s}}{a_0}\frac{x}{|x|^{n+1-s}}.$$

b) For each $x \in \mathbb{R}^n \setminus \{0\}$,

$$\lim_{\lambda \to 0^+} \lambda^{n-s} V_\delta^s(\lambda x) = \frac{c_{n,-s}}{a_0}\frac{x}{|x|^{n+1-s}}.$$

c) For any $R > 0$ there exists $M > 0$ such that for all $x \in B(0,R) \setminus \{0\}$,

$$|V_\delta^s(x)| \leq \frac{M}{|x|^{n-s}}.$$

d) For every $R_1, R_2 > 0$ there exists $M > 0$ such that for all $x \in B(0,R_1)\setminus\{0\}$ and $h \in B(0,R_2) \setminus \{x\}$,

$$|V_\delta^s(x) - V_\delta^s(x-h)| \leq M\left|\frac{x}{|x|^{n+1-s}} - \frac{x-h}{|x-h|^{n+1-s}}\right|.$$

Proof. We first prove that there exists a function $V_\delta^s \in C^\infty(\mathbb{R}^n\setminus\{0\}, \mathbb{C}^n)$ such that (6.40) holds.

We start as in the proof of [66, Proposition 2.4.8]. In order to see that V_δ^s is C^∞ away from the origin we will see that $\mathcal{F}(\hat{V}_\delta^s)(x) = \tilde{V}_\delta^s(x) = V_\delta^s(-x)$ is C^M in $\mathbb{R}^n \setminus \{0\}$ for all M. Thus, fix $M \in \mathbb{N}$ and let $\alpha \in \mathbb{N}^n$ be any multiindex such that

$$s(|\alpha| + 1) - n \geq M. \tag{6.41}$$

Now we take $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi = 1$ in $B(0,2)^c$ and $\varphi = 0$ in $B(0,1)$. Write $u = \hat{V}_\delta^s$, $u_0 = (1-\varphi)u$ and $u_\infty = \varphi u$. On the one hand, $\partial^\alpha u = \partial^\alpha u_0 + \partial^\alpha u_\infty$ in the sense of distributions and also in $\mathbb{R}^n \setminus \{0\}$. On the other hand, as u is smooth outside the origin, we have that $\partial^\alpha u_\infty$ is smooth and can calculate

$$\partial^\alpha u_\infty = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\varphi \, \partial^\beta u.$$

Write

$$v = \partial^\alpha u_0 + \sum_{\substack{\beta \leq \alpha \\ \beta \neq \alpha}} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\varphi \, \partial^\beta u. \tag{6.42}$$

Then v is a distribution with support in $B(0,2)$, so $\hat{v}$ is C^∞. Moreover, $\partial^\alpha u = v + \varphi \, \partial^\alpha u$. Thus, in order to see that $\widehat{\partial^\alpha u}$ is C^M it remains to show that $\widehat{\varphi \, \partial^\alpha u}$ is C^M. The function $\varphi \, \partial^\alpha u$ is C^∞ and, by Lemma 6.3.12,

$$|\varphi(\xi)\partial^\alpha u(\xi)| \leq \frac{C_\alpha}{1 + |\xi|^{s(|\alpha|+1)}}, \qquad \xi \in \mathbb{R}^n, \tag{6.43}$$

Having in mind (6.41), a classical result shows that $\widehat{\varphi \, \partial^\alpha u}$ is C^M.

Once we have shown that $\widehat{\partial^\alpha u}$ is C^M, we note that $\widehat{\partial^\alpha u}(\xi) = (2\pi i \xi)^\alpha \hat{u}(\xi)$. Let $\xi \in \mathbb{R}^n \setminus \{0\}$; then $\xi_j \neq 0$ for some $j \in \{1, \ldots, M\}$. Let V be a neighbourhood of ξ such that every $\eta \in V$ satisfies $\eta_j \neq 0$. Let $m \in \mathbb{N}$ be such that $s(m+1) - n \geq M$ and let α be the multiindex $(0, \ldots, 0, m, 0, \ldots, 0)$, with the component m in position j. Then α satisfies (6.41). Moreover, for any $\eta \in V$,

$$\hat{u}(\eta) = \frac{\widehat{\partial^\alpha u}(\eta)}{(2\pi i \eta_j)^m},$$

so $\hat{u}$ is of class C^M in $\mathbb{R}^n \setminus \{0\}$ for every $M \in \mathbb{N}$, and therefore, so is V_δ^s.

Once we have that V_δ^s is a function, since $\hat{V}_\delta^s$ is radial and imaginary-valued, standard properties of the Fourier transform show that V_δ^s must be radial a real-valued.

Now we show the decay of V_δ^s at infinity. Case $n=1$ is tackled in Theorem 6.3.17. So, let $n>1$, recall that

$$\hat{V_\delta^s}(\xi) = -i\frac{\xi}{|\xi|}\frac{1}{|2\pi\xi|\hat{Q}_\delta^s}.$$

We have that $V_\delta^s = \mathcal{F}^{-1}(\hat{V}_\delta^s)$, so $V_\delta^s(-x) = \mathcal{F}(\hat{V}_\delta^s)(x)$. Now, by Lemma 6.3.18

$$\begin{aligned} V_\delta^s(-x) &= \mathcal{F}\left(\hat{V_\delta^s} - \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s} + \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s}\right) \\ &= \mathcal{F}\left(\hat{V_\delta^s} - \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s}\right) + \frac{c_{n,-s}}{a_0}\frac{-x}{|x|^{n+1-s}}. \end{aligned}$$

Consequently, if we show that $\hat{V_\delta^s} - \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s}$ is in L^1, its Fourier transform will be in $C_0(\mathbb{R}^n)$. Therefore, the growth of V_δ^s around 0 will be given by that of $\frac{c_{n,-s}}{a_0}\frac{x}{|x|^{n+1+s}}$, so the desired estimated will be shown.

In order to show that $\hat{V_\delta^s} - \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s}$ is in L^1, we first notice that it is in $L^1_{\text{loc}}(\mathbb{R}^n\setminus\{0\})$, since it is in $C(\mathbb{R}^n\setminus\{0\})$. It is also in $L^1(B(0,R))$ for any $R>0$ since both $\hat{V_\delta^s}$ and $\frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s}$ are in $L^1(B(0,R))$. Hence, it remains to show the integrability of $\hat{V_\delta^s} - \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s}$ at infinity.

We will see that, in fact, the decay of $\hat{V_\delta^s} - \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s}$ at infinity is faster than any negative power of $|\xi|$. For this we observe that

$$\begin{aligned} \hat{V_\delta^s}(\xi) - \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s} &= -i\frac{\xi}{\xi}\frac{1}{|2\pi\xi|\hat{Q}_\delta^s(\xi)} - \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s} \\ &= -i\frac{\xi}{|\xi|}\frac{a_0|2\pi\xi|^{-1+s} - \hat{Q}_\delta^s(\xi)}{a_0|2\pi\xi|^s\hat{Q}_\delta^s(\xi)}. \end{aligned} \tag{6.44}$$

The terms $|2\pi\xi|^s$ and $\hat{Q}_\delta^s(\xi)$ in the denominator above only contribute as a power of $|\xi|$ in the growth at infinity (see Proposition 6.3.8). Therefore, it remains to show that the numerator above $a_0|2\pi\xi|^{-1+s} - \hat{Q}_\delta^s(\xi)$ decays faster at infinity than any negative power of $|\xi|$. Recall that $\mathcal{F}\left(\frac{1}{\gamma_\alpha|x|^{n-\alpha}}\right)(\xi) = |2\pi\xi|^{-\alpha}$. Now we consider a $\varphi\in C_c^\infty(\mathbb{R}^n)$ with $\varphi_{B(0,\frac{1}{4})}=1$ and $\varphi_{B(0,\frac{1}{2})^c}=0$. Then,

$$\begin{aligned} &a_0|2\pi\xi|^{-1+s} - \hat{Q}_\delta^s(\xi) = \\ &\mathcal{F}\left(\frac{a_0}{\gamma(1-s)|x|^{n-1+s}} - Q_\delta^s(x)\right) = \\ &\mathcal{F}\left(\frac{a_0\varphi}{\gamma(1-s)|x|^{n-1+s}} - Q_\delta^s(x)\right) + \mathcal{F}\left(\frac{a_0(1-\varphi)}{\gamma(1-s)|x|^{n-1+s}}\right). \end{aligned}$$

Looking at the expression of Q^s_δ (Lemma 6.3.2), we notice that the difference between $\frac{a_0\varphi}{\gamma(1-s)|x|^{n-1+s}}$ and $Q^s_\delta(x)$ coincide with the constant $\frac{-z_0}{\gamma(1-s)}$ in $B(0,\min\{b_0\delta,\frac{1}{4}\})$, and both have compact support. Therefore, its difference is a smooth function of compact support. In particular, it is in the Schwartz space, as well as its Fourier transform:

$$\mathcal{F}\left(\frac{a_0\varphi}{\gamma(1-s)|x|^{n-1+s}} - Q^s_\delta(x)\right) \in \mathcal{S}.$$

On the other hand, the function $\mathcal{F}\left(\frac{1-\varphi}{\gamma(1-s)|x|^{n-1+s}}\right)$ is treated in [66, Ex. 2.4.9], and it is concluded that its decay at infinity is faster than any negative power of $|\xi|$.

We have concluded that there exists $W \in C_0(\mathbb{R}^n,\mathbb{R}^n)$ such that

$$V^s_\delta(x) = W(x) + \frac{c_{n,-s}}{a_0}\frac{x}{|x|^{n+1-s}}.$$

With this we have that for each $x \in \mathbb{R}^n \setminus \{0\}$, $h \in \mathbb{R}^n \setminus \{x\}$ and $\lambda > 0$,

$$\lambda^{n-s}V^s_\delta(\lambda x) = \lambda^{n-s}W(\lambda x) + \frac{c_{n,-s}}{a_0}\frac{x}{|x|^{n+1-s}},$$

so

$$\lim_{\lambda\to 0^+}\lambda^{n-s}V^s_\delta(\lambda x) = \frac{c_{n,-s}}{a_0}\frac{x}{|x|^{n+1-s}}.$$

We also have that given $R > 0$, for all $x \in B(0,R) \setminus \{0\}$,

$$|V^s_\delta(x)| \le \|W\|_{\infty,B(0,R)} + \frac{c_{n,-s}}{a_0}\frac{1}{|x|^{n-s}} \le \left(\|W\|_\infty R^{n-s} + \frac{c_{n,-s}}{a_0}\right)\frac{1}{|x|^{n-s}}.$$

As for the inequality *d)*, we first prove that W is Lipschitz. We know that W is $C^\infty(\mathbb{R}^n \setminus \{0\})$. In addition, the proof above shows that $W(x) = \mathcal{F}(Z)(-x)$ with $Z(\xi) := \hat{V}^s_\delta(\xi) - \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s}$, whose decay at infinity is faster than any negative power of $|\xi|$. Thus, $\nabla W(x) = \mathcal{F}(2\pi i\xi Z(\xi))(-x)$. Since $2\pi i\xi Z(\xi)$ is integrable then $\mathcal{F}(2\pi i\xi Z(\xi))$ is bounded, so W is Lipschitz.

Once we know that W is Lipschitz, we estimate

$$\begin{aligned}|V^s_\delta(x) - V^s_\delta(x-h)| &\le \|DW\|_{L^\infty(\mathbb{R}^n)}|h| + \frac{|c_{n,-s}|}{a_0}\left|\frac{x}{|x|^{n+1-s}} - \frac{x-h}{|x-h|^{n+1-s}}\right| \\ &\le M\left|\frac{x}{|x|^{n+1-s}} - \frac{x-h}{|x-h|^{n+1-s}}\right|,\end{aligned}$$

for a suitable constant $M > 0$ coming from Lemma 6.3.13. The proof is complete. $\square$

Remark 6.3.1. *It can also be proved (and easy to see heuristically) that for each $x \in \mathbb{R}^n \setminus \{0\}$,*

$$\lim_{\lambda\to\infty} \lambda^{n-1} V_\delta^s(\lambda x) = \frac{c_{n,-1}}{\|Q_\delta^s\|_{L^1}} \frac{x}{|x|^n}.$$

Nevertheless it is not going to be needed in this work.

Now we can provide the proof of Proposition 6.3.5.

Proof of Proposition 6.3.5. Given the function V_δ^s from Theorem 6.3.14 we want to check the equality

$$\int_{\mathbb{R}^n} V_\delta^s(z) Q_\delta^s(y-z)dz = \frac{1}{\sigma_{n-1}} \frac{y}{|y|^n}, \qquad y \in \mathbb{R}^n.$$

In order to do so, we are going to see the equality of the Fourier transforms of both terms. Notice that V_δ^s and Q_δ^s can be seen as tempered distributions, and, in particular, Q_δ^s with compact support. Hence, by Lemmas 6.3.20 and 6.3.19 we have that the desired equality is equivalent to

$$\hat{V}_\delta^s(\xi)\hat{Q}_\delta^s(\xi) = -i\frac{\xi}{|\xi|}\frac{1}{|2\pi\xi|}$$

which holds by (6.40) and the result follows. □

Technical results

The next lemma is used in Proposition 6.3.10.

Lemma 6.3.15. *Let $0 < s < 1$ and $0 < \delta$. Then*

$$\nabla Q_\delta^s \chi_{B(0,\varepsilon)^c} \to \nabla Q_\delta^s \quad \textit{in } \mathcal{S}' \quad \textit{as } \varepsilon \to 0.$$

Proof. We recall that $\nabla Q_\delta^s(x) = -(n+s-1)\frac{\rho_\delta(x)}{|x|}\frac{x}{|x|}$. Fix $j \in \{1, \dots, n\}$: we shall prove the desired convergence for the j-th component of ∇Q_δ^s. Let $\varphi \in \mathcal{S}$. Using the notation $B_j^{\pm}(0,\varepsilon)^c = \{x \in B(0,\varepsilon)^c : \pm x_j > 0\}$ and $\bar{w}_\delta$ for the radial representation of w_δ we have

$$\begin{aligned}\int_{B(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} \bar{w}_\delta(x)\varphi(x)\,dx &= \int_{B_j^-(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} \bar{w}_\delta(x)\varphi(x)\,dx \\ &\quad + \int_{B_j^+(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} \bar{w}_\delta(x)\varphi(x)\,dx \\ &= \int_{B_j^+(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} \bar{w}_\delta(x)(\varphi(x)-\varphi(-x))\,dx.\end{aligned}$$

By the mean value theorem,

$$\left|\frac{x_j}{|x|^{n+s+1}}\bar{w}_\delta(x)(\varphi(x)-\varphi(-x))\chi_{B_j^+(0,\varepsilon)^c}(x)\right| \leq \frac{2\|\nabla\varphi\|_\infty\|\bar{w}_\delta\|_\infty}{|x|^{n+s-1}}\chi_{B(0,\delta)}(x).$$

By dominated convergence we obtain that

$$\int_{B_j^+(0,\varepsilon)^c}\frac{x_j}{|x|^{n+s+1}}\bar{w}_\delta(x)(\varphi(x)-\varphi(-x))\,dx$$

converges to

$$\int_{\{x_j>0\}}\frac{x_j}{|x|^{n+s+1}}\bar{w}_\delta(x)(\varphi(x)-\varphi(-x))\,dx$$

as $\varepsilon \to 0$. This proves that

$$\int_{B(0,\varepsilon)^c}\frac{x_j}{|x|^{n+s+1}}\bar{w}_\delta(x)\varphi(x)\,dx \to \int \frac{x_j}{|x|^{n+s+1}}\bar{w}_\delta(x)\varphi(x)\,dx$$

and the conclusion follows. □

1-dimensional case

In this appendix we treat the particular case of determining V_δ^s when $n=1$. Before doing so we provide an auxiliary result necessary to prove the following theorem.

Lemma 6.3.16. *Let $0<s<1$, $\delta>0$ and $n=1$. Then the function $Z(\xi) := \hat{V}_\delta^s(\xi) - \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s}$ can be identified with the tempered distribution*

$$\langle Z,\varphi\rangle = \int_0^\infty Z(\xi)(\varphi(\xi)-\varphi(-\xi))\,d\xi, \qquad \varphi\in\mathcal{S} \tag{6.45}$$

and we have the convergence

$$Z\chi_{B(0,\varepsilon)^c} \to Z \quad \text{in } \mathcal{S}' \quad \text{as } \varepsilon\to 0. \tag{6.46}$$

Proof. Let us see that formula (6.45) defines a tempered distribution. By Propositions 6.3.8 and 6.3.10, there exists $C>0$ such that

$$|Z(\xi)| \leq \frac{1}{2\pi|\xi|}\frac{1}{\hat{Q}_\delta^s(\xi)} + \frac{1}{a_0}\frac{1}{|2\pi\xi|^s} \leq \frac{C}{|\xi|}, \qquad |\xi|\leq 1.$$

Thus, by the mean value theorem

$$\left|\int_0^1 Z(\xi)(\varphi(\xi)-\varphi(-\xi))\,d\xi\right| \leq 2C\|\varphi'\|_\infty. \tag{6.47}$$

On the other hand, in the proof of Theorem 6.3.14, and, concretely, (6.44), we saw that Z decays to 0 at infinity faster than any negative power of ξ. In particular,

$$|Z(\xi)| \le \frac{C}{1+|\xi|^2}, \qquad |\xi| \ge 1.$$

Consequently,

$$\left| \int_1^\infty Z(\xi)(\varphi(\xi) - \varphi(-\xi))\, d\xi \right| \le 2C \int_1^\infty \frac{1}{1+\xi^2}\, d\xi\, \|\varphi\|_\infty. \tag{6.48}$$

Estimates (6.47) and (6.48) show that Z defined by (6.45) is in $\mathcal{S}'$.

As explained before, the fact that Z decays to 0 at infinity faster than any negative power of ξ implies that $Z\chi_{B(0,\varepsilon)^c} \in L^1(\mathbb{R})$ for all $\varepsilon > 0$. In particular, $Z\chi_{B(0,\varepsilon)^c}$ considered as a distribution acts as follows: for each $\varphi \in \mathcal{S}$,

$$\langle Z\chi_{B(0,\varepsilon)^c}, \varphi\rangle = \int_{B(0,\varepsilon)^c} Z(\xi)\, \varphi(\xi)\, d\xi = \int_\varepsilon^\infty Z(\xi)(\varphi(\xi) - \varphi(-\xi))\, d\xi.$$

As we saw above, the function $Z(\xi)(\varphi(\xi) - \varphi(-\xi))$ is in $L^1((0,\infty))$, so, by dominated convergence we have that

$$\int_\varepsilon^\infty Z(\xi)(\varphi(\xi) - \varphi(-\xi))\, d\xi \to \int_0^\infty Z(\xi)(\varphi(\xi) - \varphi(-\xi))\, d\xi \qquad \text{as } \varepsilon \to 0,$$

which justifies the identification of the function Z with the distribution (6.45) and shows the convergence (6.46). □

The following is the one-dimensional version of Theorem 6.3.14.

Theorem 6.3.17. *Let $0 < s < 1$, $0 < \delta$ and $n = 1$. Then there exists a real-valued function $V_\delta^s \in C^\infty(\mathbb{R} \setminus \{0\})$ whose Fourier transform is given by*

$$\hat{V}_\delta^s(\xi) = \frac{-i\xi}{2\pi|\xi|^2} \frac{1}{\hat{Q}_\delta^s(\xi)}.$$

Actually, there exists $W \in C_b(\mathbb{R})$ such that

$$V_\delta^s(x) = W(x) + \frac{c_{1,-s}}{a_0} \frac{x}{|x|^{2-s}}. \tag{6.49}$$

In addition, $V_\delta^s \in L^1_{\mathrm{loc}}(\mathbb{R})$ and

a) $\lim_{|x|\to\infty} \operatorname{sgn}(x) V_\delta^s(x) = \frac{1}{2\|Q_\delta^s\|_{L^1(\mathbb{R})}}$, *where* sgn *is the sign function.*

b) $\lim_{x\to 0} \operatorname{sgn}(x)|x|^{1-s} V_\delta^s(x) = \frac{c_{1,-s}}{a_0}$.

c) For any $R > 0$ *there exists* $M > 0$ *such that for all* $x \in B(0,R) \setminus \{0\}$,

$$|V_\delta^s(x)| \le \frac{M}{|x|^{1-s}}.$$

d) For every $R_1, R_2 > 0$ *there exists* $M > 0$ *such that for all* $x \in B(0,R_1)\setminus\{0\}$ *and* $h \in B(0,R_2) \setminus \{x\}$,

$$|V_\delta^s(x) - V_\delta^s(x-h)| \le M \left| \frac{x}{|x|^{2-s}} - \frac{x-h}{|x-h|^{2-s}} \right|.$$

Proof. The existence of $V_\delta^s \in C^\infty(\mathbb{R}\setminus\{0\})$ as the Fourier transform of $\hat{V}_\delta^s$ was given by the first part of Theorem 6.3.14, which is valid for $n = 1$. Thus, we start with (6.49).

As in Theorem 6.3.14, we compute the Fourier transform in the following way using Lemma 6.3.18

$$\begin{aligned} V(-x) &= \mathcal{F}\left(\hat{V}_\delta^s - \frac{-i\xi}{a_0|\xi|} \frac{1}{|2\pi\xi|^s} + \frac{-i\xi}{a_0|\xi|} \frac{1}{|2\pi\xi|^s} \right)(x) \\ &= \mathcal{F}\left(\hat{V}_\delta^s - \frac{-i\xi}{a_0|\xi|} \frac{1}{|2\pi\xi|^s} \right)(x) + \frac{c_{n,-s}}{a_0} \frac{-x}{|x|^{2-s}} \end{aligned}$$

where, despite $\hat{V}_\delta^s$ not being integrable around zero, we are going to see that the function

$$\tilde{W}(x) := \int_{\mathbb{R}} \left(-\frac{i\xi}{2\pi|\xi|^2} \frac{1}{\hat{Q}_\delta^s(\xi)} - \frac{-i\xi}{a_0|\xi|} \frac{1}{|2\pi\xi|^s} \right)(-i)\sin(2\pi\xi x)\, d\xi \qquad (6.50)$$

is well defined as a Lebesgue integral and is the Fourier transform of $Z(\xi) := \hat{V}_\delta^s(\xi) - \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s}$. Actually, the same argument used in Theorem 6.3.14 *a)* (see the argument following (6.44)) shows that $Z(\xi)$ goes to 0 at infinity faster than any negative power of ξ; consequently, we just have to focus on the local integrability of the integrand in (6.50), which is in fact given by the inequality

$$\begin{aligned} \left| \left(-\frac{i\xi}{2\pi|\xi|^2} \frac{1}{\hat{Q}_\delta^s(\xi)} - \frac{-i\xi}{a_0|\xi|} \frac{1}{|2\pi\xi|^s} \right)(-i)\sin(2\pi\xi x) \right| &\le \left| \frac{\sin(2\pi\xi x)}{2\pi\xi \hat{Q}_\delta^s(\xi)} \right| + \frac{1}{a_0|2\pi\xi|^s} \\ &\le \frac{|x|}{\hat{Q}_\delta^s(\xi)} + \frac{1}{a_0|2\pi\xi|^s}, \end{aligned} \qquad (6.51)$$

with the right-hand side being locally integrable in ξ. This gives us the Lebesgue integrability of the integrand in (6.50). With respect to the Fourier transform of Z we argue as in Proposition 6.3.10 with equation (6.24). As explained in Lemma 6.3.16, $Z\chi_{B(0,\varepsilon)^c} \in L^1(\mathbb{R})$ for all $\varepsilon > 0$ and Z is considered as a tempered distribution via formula (6.45). Moreover,

$$Z\chi_{B(0,\varepsilon)^c} \to Z \quad \text{in } \mathcal{S}' \quad \text{as } \varepsilon \to 0,$$

so, by the continuity of the Fourier transform in $\mathcal{S}'$

$$\mathcal{F}\left(Z\chi_{B(0,\varepsilon)^c}\right) \to \mathcal{F}\left(Z\right) \quad \text{in } \mathcal{S}' \quad \text{as } \varepsilon \to 0.$$

We now compute

$$\begin{aligned}\mathcal{F}\left(Z\chi_{B(0,\varepsilon)^c}\right)(x) &= \int_{B(0,\varepsilon)^c}\left(-\frac{i\xi}{2\pi|\xi|^2}\frac{1}{\hat{Q}^s_\delta(\xi)} - \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s}\right)e^{-2\pi i\xi x}\,d\xi \\ &= \int_{B(0,\varepsilon)^c}\left(-\frac{i\xi}{2\pi|\xi|^2}\frac{1}{\hat{Q}^s_\delta(\xi)} - \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s}\right)(-i)\sin(2\pi\xi x)\,d\xi,\end{aligned}$$

where we have used the odd symmetry. Now, by (6.51) and the decay of Z, we have that $|Z(\xi)\sin(2\pi\xi x)|$ is integrable in $\xi \in \mathbb{R}$. Therefore,

$$\int_{B(0,\varepsilon)^c}\left(-\frac{i\xi}{2\pi|\xi|^2}\frac{1}{\hat{Q}^s_\delta(\xi)} - \frac{-i\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s}\right)(-i)\sin(2\pi\xi x)\,d\xi$$

converges to

$$\int -iZ(\xi)\sin\left(2\pi\xi x\right)d\xi, \qquad \text{as } \varepsilon \to 0.$$

This proves that $\tilde{W} = \mathcal{F}(Z)$ and so, we finally take $W(x) = \tilde{W}(-x)$.

The function W is $C^\infty(\mathbb{R}\setminus\{0\})$ as a difference of two $C^\infty(\mathbb{R}\setminus\{0\})$ functions. Let us see that W is continuous, so let us check its continuity at 0. From (6.50) we have the formula

$$W(x) = \int\left(\frac{\xi}{2\pi|\xi|^2}\frac{1}{\hat{Q}^s_\delta(\xi)} - \frac{\xi}{a_0|\xi|}\frac{1}{|2\pi\xi|^s}\right)\sin(2\pi\xi x)\,d\xi.$$

For $|x| \leq 1$, we have the bound

$$\left| \left(\frac{\xi}{2\pi|\xi|^2} \frac{1}{\hat{Q}^s_\delta(\xi)} - \frac{\xi}{a_0|\xi|} \frac{1}{|2\pi\xi|^s} \right) \sin(2\pi\xi x) \right| \leq$$
$$\left(\left| \frac{\sin(2\pi\xi x)}{2\pi\xi\hat{Q}^s_\delta(\xi)} \right| + \frac{1}{a_0|2\pi\xi|^s} \right) \chi_{B(0,1)}(\xi) + |Z(\xi)|\chi_{B(0,1)^c}(\xi) \leq$$
$$\left(\frac{1}{\hat{Q}^s_\delta(\xi)} + \frac{1}{a_0|2\pi\xi|^s} \right) \chi_{B(0,1)}(\xi) + |Z(\xi)|\chi_{B(0,1)^c}(\xi)$$

with

$$\int_{B(0,1)} \left(\frac{1}{\hat{Q}^s_\delta(\xi)} + \frac{1}{a_0|2\pi\xi|^s} \right) d\xi + \int_{B(0,1)^c} |Z(\xi)|\, d\xi < \infty,$$

which allows us to use dominated convergence and obtain that $\lim_{x\to 0} W(x) = 0 = W(0)$.

Now we prove that W is bounded. To that end, we study its behaviour at infinity, for which it is useful to introduce the function

$$Y(\xi) = -\frac{i\xi}{2\pi|\xi|^2} \frac{1}{\hat{Q}^s_\delta(0)} \chi_{B(0,1)}(\xi)$$

and express

$$\tilde{W} = \mathcal{F}(Z\chi_{B(0,1)} - Y) + \mathcal{F}(Y) + \mathcal{F}(Z\chi_{B(0,1)^c}).$$

Since $Z\chi_{B(0,1)^c} \in L^1(\mathbb{R})$, by the Riemann–Lebesgue Lemma $\mathcal{F}(Z\chi_{B(0,1)^c}) \in C_0(\mathbb{R})$. Now we study the function

$$Z\chi_{B(0,1)}(\xi) - Y(\xi) = \left[-\frac{i\xi}{2\pi|\xi|^2} \left(\frac{1}{\hat{Q}^s_\delta(\xi)} - \frac{1}{\hat{Q}^s_\delta(0)} \right) - \frac{-i\xi}{a_0|\xi|} \frac{1}{|2\pi\xi|^s} \right] \chi_{B(0,1)}(\xi).$$

Now, for $\xi \in B(0,1)$, by the mean value theorem,

$$\left| -\frac{i\xi}{2\pi|\xi|^2} \left(\frac{1}{\hat{Q}^s_\delta(\xi)} - \frac{1}{\hat{Q}^s_\delta(0)} \right) \right| \leq \frac{1}{2\pi} \sup_{B(0,1)} \frac{\left(\hat{Q}^s_\delta\right)'}{\left(\hat{Q}^s_\delta\right)^2} < \infty,$$

and, on the other hand,

$$\left| \frac{-i\xi}{a_0|\xi|} \frac{1}{|2\pi\xi|^s} \right| \leq \frac{1}{a_0} \frac{1}{|2\pi\xi|^s},$$

which is integrable in $B(0,1)$. Therefore, $Z\chi_{B(0,1)}-Y \in L^1(\mathbb{R})$, so $\mathcal{F}(Z\chi_{B(0,1)}-Y) \in C_0(\mathbb{R})$. Consequently, the limit at infinity of $\tilde{W}$ is the same as that of $\mathcal{F}(Y)$, which we analyze now. By odd symmetry,

$$\mathcal{F}(Y)(-x) = \frac{1}{\pi\hat{Q}_\delta^s(0)} \int_0^1 \frac{1}{\xi}\sin(2\pi\xi x)\,d\xi = \frac{1}{\pi\hat{Q}_\delta^s(0)} \int_0^x \frac{1}{\xi}\sin(2\pi\xi)\,d\xi.$$

This latter function is known to be bounded, and, in fact,

$$\int_0^\infty \frac{1}{\xi}\sin(2\pi\xi)\,d\xi = \frac{\pi}{2},$$

so

$$\lim_{x\to+\infty} W(x) = \lim_{x\to+\infty} \tilde{W}(-x) = \lim_{x\to+\infty} \mathcal{F}(Y)(-x) = \frac{1}{2\hat{Q}_\delta^s(0)} = \frac{1}{2\|Q_\delta^s\|_{L^1(\mathbb{R}^n)}}.$$

This proves that W is bounded and also limit *a)*.

Combining (6.49) and $W \in C_b(\mathbb{R})$ we have limit *b)*. This limit and the fact $V_\delta^s \in C^\infty(\mathbb{R}\setminus\{0\})$ implies property *c)*.

Part *d)* is proved with the same argument as in Theorem 6.3.14. □

Fourier analysis auxiliary results

In this appendix we collect together several Fourier analysis results needed throughout the chapter. First, we recall the following definitions and properties.

Remark 6.3.2. *Let $u, v \in \mathcal{S}'$. For each $f : \mathbb{R}^n \to \mathbb{R}$, we define $\tilde{f}(x) := f(-x)$ for every $x \in \mathbb{R}^n$.*

a) $\tilde{v} \in \mathcal{S}'$ is defined as

$$\langle \tilde{v}, \varphi\rangle = \langle v, \tilde{\varphi}\rangle \quad \forall\, \varphi \in \mathcal{S}.$$

b) Assume that $\tilde{v} * \varphi \in \mathcal{S}$ for every $\varphi \in \mathcal{S}$. Then the tempered distribution $v * u$ is defined as

$$\langle v * u, \varphi\rangle = \langle u, \tilde{v} * \varphi\rangle \quad \forall\, \varphi \in \mathcal{S}.$$

c) We have that $\mathcal{F}(\hat{v}) = \tilde{v}$.

We now compute the Fourier transform of the vectorial version of the Riesz potential.

Lemma 6.3.18. *a) Let $n \geq 2$, $0 < \alpha < n-1$ and $j \in \{1, \dots, n\}$. Then*

$$\frac{n-\alpha-1}{\gamma(1+\alpha)} \widehat{\frac{x_j}{|x|^{n-\alpha+1}}}(\xi) = -i\frac{\xi_j}{|\xi|}|2\pi\xi|^{-\alpha} = -i\frac{\xi_j}{|\xi|}\hat{I}_\alpha. \tag{6.52}$$

b) If $n = 1$ and $0 < s < 1$, then

$$c_{1,-s}\widehat{\frac{x}{|x|^{2-s}}} = -\frac{i\xi}{|\xi|}\frac{1}{|2\pi\xi|^s}.$$

Proof. On the one hand, we have that

$$\frac{1}{\gamma(1+\alpha)}\frac{\partial}{\partial x_j}\frac{1}{|x|^{n-(\alpha+1)}} = -\frac{n-\alpha-1}{\gamma(1+\alpha)}\frac{x_j}{|x|^{n-\alpha+1}}.$$

Thus,

$$\frac{1}{\gamma(1+\alpha)}\left(\frac{\partial}{\partial x_j}\widehat{\frac{1}{|x|^{n-(\alpha+1)}}}\right)(\xi) = -\frac{n-\alpha-1}{\gamma(1+\alpha)}\widehat{\frac{x_j}{|x|^{n-\alpha+1}}}(\xi), \tag{6.53}$$

whereas, on the other hand, by standard properties of the Fourier transform,

$$\frac{1}{\gamma(1+\alpha)}\left(\frac{\partial}{\partial x_j}\widehat{\frac{1}{|x|^{n-(\alpha+1)}}}\right)(\xi) = 2\pi i\xi_j\hat{I}_{1+\alpha} = 2\pi i\xi_j|2\pi\xi|^{-(1+\alpha)} = i\frac{\xi_j}{|\xi|}|2\pi\xi|^{-\alpha}. \tag{6.54}$$

Putting together (6.53) and (6.54) we obtain the conclusion of *a)*.

Now we present the proof of *b)*. We recall the formula of the fractional version of the fundamental theorem of Calculus (Theorem 3.5.1) (see also [38, Th. 3.11], [99, Th. 1.12] or [92, Prop. 15.8]), where, for every $u \in C_c^\infty(\mathbb{R})$,

$$u(x) = c_{1,-s}\int D^s u(y)\frac{x-y}{|x-y|^{2-s}}dy.$$

Next, we take Fourier transform and use the formula for the convolution of a distribution with a Schwartz function:

$$\hat{u}(\xi) = \widehat{D^s u}(\xi)\mathcal{F}\left(c_{1,-s}\frac{x}{|x|^{2-s}}\right) = \frac{2\pi i\xi}{|2\pi\xi|}|2\pi\xi|^s\hat{u}(\xi)\mathcal{F}\left(c_{1,-s}\frac{x}{|x|^{2-s}}\right)$$

where we have used the Fourier transform of $D^s u$ (see Lemma 3.1.7). Now, we multiply both terms by $-i2\pi\xi$ and obtain that

$$-i2\pi\xi\hat{u}(\xi) = |2\pi\xi|^{1+s}\hat{u}(\xi)\mathcal{F}\left(c_{1,-s}\frac{x}{|x|^{2-s}}\right).$$

Since that equality holds for every $u \in C_c^\infty(\mathbb{R})$, the statement follows. $\square$

The following Fourier transform is obtained.

Lemma 6.3.19. *The following equality holds:*

$$\mathcal{F}\left(\frac{1}{\sigma_{n-1}}\frac{x}{|x|^n}\right)(\xi) = -i\frac{\xi}{|\xi|}\frac{1}{|2\pi\xi|}.$$

Proof. Since $\frac{1}{\sigma_{n-1}}\frac{x}{|x|^n} \in L^1(B(0,1)) + L^\infty(B(0,1)^c)$ we have that $\frac{1}{\sigma_{n-1}}\frac{x}{|x|^n} \in \mathcal{S}'$, and so does its Fourier transform. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. We apply the Fourier transform to the representation formula of Proposition 6.3.1, obtaining that

$$\hat{\varphi}(\xi) = \widehat{\nabla\varphi}(\xi)\cdot\mathcal{F}\left(\frac{x}{\sigma_{n-1}|x|^n}\right)(\xi) = 2\pi i\xi\hat{\varphi}(\xi)\cdot\mathcal{F}\left(\frac{x}{\sigma_{n-1}|x|^n}\right)(\xi).$$

Since this is true for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ we infer that

$$1 = 2\pi i\xi\cdot\mathcal{F}\left(\frac{x}{\sigma_{n-1}|x|^n}\right)(\xi).$$

Therefore, there exists a function $G : \mathbb{R}^n \to \mathbb{R}^n$ such that $\xi\cdot G(\xi) = 0$ and

$$\mathcal{F}\left(\frac{x}{\sigma_{n-1}|x|^n}\right)(\xi) = -i\frac{\xi}{|\xi|}\frac{1}{|2\pi\xi|} + G(\xi).$$

On the other hand, $\mathcal{F}\left(\frac{x}{\sigma_{n-1}|x|^n}\right)$ must be a vector radial function, as the Fourier transform of a vector radial function. Consequently (recall Definition 6.1.5), there exists $g : \mathbb{R}^n \to \mathbb{R}$ such that $G(\xi) = \xi\, g(\xi)$. Thus, $|\xi|^2 g(\xi) = 0$, so $g = 0$ and, hence, $G = 0$ a.e. The proof is concluded. □

Lemma 6.3.20. *Let $V, Q \in \mathcal{S}'$ be such that Q is a distribution with compact support. Then*

$$\widehat{V * Q} = \hat{V}\hat{Q}.$$

Proof. Firstly, we recall that the convolution $V * Q$ is well defined since $\tilde{Q}*\varphi \in \mathcal{S}$ for every $\varphi \in \mathcal{S}$ [66, Theorem 2.3.20], and its action is defined as

$$\langle V * Q, \varphi\rangle = \langle V, \varphi * \tilde{Q}\rangle \quad \text{for every } \varphi \in \mathcal{S},$$

where we are using the notation $\tilde{Q}$ from Remark 6.3.2. Now, by definition of the Fourier transform in the sense of distributions, we have that, for every $\varphi \in \mathcal{S}$,

$$\langle \widehat{V * Q}, \varphi\rangle = \langle V * Q, \hat{\varphi}\rangle = \langle V, \hat{\varphi} * \tilde{Q}\rangle. \tag{6.55}$$

Next, by the Fourier transform of a convolution (of a distribution times a Schwartz function),

$$\hat{\varphi} * \tilde{Q} = \mathcal{F}(\varphi \mathcal{F}^{-1}(\tilde{Q})) = \mathcal{F}(\varphi \hat{Q}),$$

since $\mathcal{F}^{-1}(\tilde{Q}) = \hat{Q}$ (Remark 6.3.2). This also tells us that $\varphi\hat{Q}$ belongs to $\mathcal{S}$ because so does $\hat{\varphi} * \tilde{Q}$ (by the bijection of the Fourier transform in $\mathcal{S}$). Actually, it is known that $\hat{Q}$ is a smooth function (see [66, Theorem 2.3.21]). Therefore, continuing with (6.55) and using again the duality of the Fourier transform,

$$\langle \widehat{V * Q}, \varphi \rangle = \langle V, \hat{\varphi} * \tilde{Q} \rangle = \langle V, \mathcal{F}(\varphi \hat{Q}) \rangle = \langle \hat{V}, \varphi \hat{Q} \rangle.$$

As $\varphi\hat{Q} \in \mathcal{S}$, the product $\hat{V}\hat{Q}$ is well defined in a distributional sense and

$$\langle \hat{V}\hat{Q}, \varphi \rangle = \langle \hat{V}, \varphi \hat{Q} \rangle = \langle \widehat{V * Q}, \varphi \rangle.$$

As a consequence, the statement holds. □

6.4 Nonlocal Poincarè and Sobolev inequalities and Compact Embedding

In this section we will use this nonlocal fundamental theorem of calculus (Theorem 6.3.6) to prove compact embeddings of the spaces $H^{s,p,\delta}$ with a complementary-value condition into L^q spaces, as well as a Poincaré inequality for functions in the space $H^{s,p,\delta}(\Omega)$ which vanishes in a tubular neighbourhood of the boundary.

Recall the set $\Omega_{-\delta} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$. We define the subspace $H_0^{s,p,\delta}(\Omega_{-\delta})$ as the closure of $C_c^\infty(\Omega_{-\delta})$ in $H^{s,p,\delta}(\Omega)$:

$$H_0^{s,p,\delta}(\Omega_{-\delta}) = \overline{C_c^\infty(\Omega_{-\delta})}^{H^{s,p,\delta}(\Omega)}.$$

It is immediate to check that any $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$ satisfies $u = 0$ a.e. in $\Omega_\delta \setminus \Omega_{-\delta}$. In addition, given $g \in H^{s,p,\delta}(\Omega)$ we define the affine subspace $H_g^{s,p,\delta}(\Omega_{-\delta})$ as $g + H_0^{s,p,\delta}(\Omega_{-\delta})$. In this section we will use several times the observation that $\operatorname{supp} D_\delta^s u \subset \operatorname{supp} u + B(0,\delta)$.

Next we prove the Poincaré–Sobolev inequality in $H_0^{s,p,\delta}(\Omega_{-\delta})$. This result is known in the fractional case, i.e., for the space $H^{s,p}(\mathbb{R}^n)$, which is stated in Theorem 3.5.3 (see [99, Th. 1.8] or [21, Theorem 2.2]).

Theorem 6.4.1. *Let $0 < s < 1$, $\delta > 0$ and $1 < p < \infty$ with $sp < n$. Then, there exists $C > 0$ such that for all $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$,*

$$\|u\|_{L^q(\Omega)} \leq C \|D_\delta^s u\|_{L^p(\Omega)}$$

for every $q \in [1, p_s^*]$, where $p_s^* = \frac{np}{n-sp}$.

Proof. By density, it is enough to prove the inequality for $u \in C_c^\infty(\Omega_\delta)$.

Fix $x \in \Omega$ and let $C > 0$ denote a constant whose value may vary through this process. Notice that by Proposition 6.3.3, $D_\delta^s u \in C^\infty(\mathbb{R}^n)$ and since $u = 0$ in $\Omega_\delta \setminus \Omega_{-\delta}$, we have $\operatorname{supp} D_\delta^s u \subset \Omega$. By Theorem 6.3.6 and Proposition 6.3.5,

$$|u(x)| \leq \int_\Omega |D_\delta^s u(y)||V_\delta^s(x-y)|\, dy \leq C \int_\Omega \frac{|D_\delta^s u(y)|}{|x-y|^{n-s}}\, dy = C\left(I_s * |D_\delta^s u|\right)(x).$$

On the other hand, by the Hardy–Littlewood–Sobolev inequality we have that

$$\|I_s * |D_\delta^s u|\|_{L^{p_s^*}(\mathbb{R}^n)} \leq C\|D_\delta^s u\|_{L^p(\mathbb{R}^n)}.$$

Therefore, for every $q \in [1, p_s^*]$, using that $\operatorname{supp} u \subset \Omega_{-\delta}$,

$$\begin{aligned} \|u\|_{L^q(\Omega)} &\leq C\|u\|_{L^{p_s^*}(\Omega)} \leq C\|I_s * |D_\delta^s u|\|_{L^{p_s^*}(\mathbb{R}^n)} \leq C\|D_\delta^s u\|_{L^p(\mathbb{R}^n)} \\ &= C\|D_\delta^s u\|_{L^p(\Omega)}. \end{aligned}$$

□

As a corollary, it is obtained a nonlocal Poincaré inequality.

Theorem 6.4.2. *Let $0 < s < 1$, $0 < \delta$, and $1 < p < \infty$. Then there exists $C > 0$ such that*

$$\|u\|_{L^p(\Omega)} \leq C \|D_\delta^s u\|_{L^p(\Omega)} \quad \forall\, u \in H_0^{s,p,\delta}(\Omega_{-\delta}).$$

Proof. If $sp < n$, the result is a particular case of Theorem 6.4.1. If $sp \geq n$ and $n \geq 2$, we take $q = \frac{np}{n+sp}$, which satisfies

$$1 < q < p, \qquad sq < n \quad \text{and} \quad q_s^* = p.$$

By Theorem 6.4.1, we have the inequality

$$\|u\|_{L^p(\Omega)} = \|u\|_{L^{q^*}(\Omega)} \leq C\|D_\delta^s u\|_{L^q(\Omega)} \leq C\|D_\delta^s u\|_{L^p(\Omega)}.$$

If $sp \geq n$ and $n = 1$ we take any q satisfying

$$1 < q < p, \qquad sq < n \quad \text{and} \quad q_s^* \geq p,$$

which is easily seen to exist. The proof is concluded analogously. □

Next we introduce a nonlocal analogue of Morrey inequality, whose fractional version was shown in Theorem 3.5.7, [99, Theorem 1.11].

Theorem 6.4.3. *Let $0 < \delta$, $0 < s < 1$ and $1 < p < \infty$ be such that $sp > n$. Then there exists $C > 0$ such that for all $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$*

$$|u(x) - u(y)| \le C|x-y|^{s-\frac{n}{p}} \|D_\delta^s u\|_{L^p(\Omega)}, \qquad a.e.\ x, y \in \Omega.$$

In addition, any $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$ has a representative which is Hölder continuous of exponent $s - \frac{n}{p}$, and the continuous inclusion $H_0^{s,p,\delta}(\Omega_{-\delta}) \subset C^{0,s-\frac{n}{p}}(\Omega)$ holds.

Proof. Let $C = C(n, s, \delta, p, |\Omega|)$ denote a constant whose value may vary through the different steps.

By a standard density argument, it is enough to prove that

$$|u(x) - u(y)| \le C|x-y|^{s-\frac{n}{p}} \|D_\delta^s u\|_{L^p(\Omega)}, \qquad x, y \in \Omega$$

for all $u \in C_c^\infty(\Omega_{-\delta})$. Fix $x, y \in \Omega$. By Theorem 6.3.6, and later by Theorem 6.3.14 *d)* there exists $C > 0$ such that

$$\begin{aligned} |u(x) - u(y)| &= \left| \int_{\mathbb{R}^n} D_\delta^s u(z) V_\delta^s(x-z) dz - \int_{\mathbb{R}^n} D_\delta^s u(z) V_\delta^s(y-z) dz \right| \\ &\le \int_\Omega |V_\delta^s(x-z) - V_\delta^s(y-z)| \, |D_\delta^s u(z)| dz \\ &\le C \int_\Omega \left| \frac{x-z}{|x-z|^{n+1-s}} - \frac{y-z}{|y-z|^{n+1-s}} \right| |D_\delta^s u(z)| dz. \end{aligned} \tag{6.56}$$

Now define $r := |x-y|$. We have

$$\begin{aligned} |u(x) - u(y)| \le & C \int_{B(x,2r)} |x-z|^{s-n} \, |D_\delta^s u(z)| dz \\ & + C \int_{B(x,2r)} |y-z|^{s-n} \, |D_\delta^s u(z)| dz \\ & + C \int_{B(x,2r)^c} \left| \frac{x-z}{|x-z|^{n+1-s}} - \frac{y-z}{|y-z|^{n+1-s}} \right| |D_\delta^s u(z)| dz. \end{aligned} \tag{6.57}$$

For the first term we have that by Hölder's inequality,

$$\int_{B(x,2r)} |x-z|^{s-n} |D^s_\delta u(z)| dz \leq$$
$$\left(\int_{B(x,2r)} |x-z|^{(s-n)p'} dz\right)^{\frac{1}{p'}} \left(\int_{B(x,2r)} |D^s_\delta u(z)|^p dz\right)^{\frac{1}{p}} \leq \tag{6.58}$$
$$2^{1-\frac{n}{p}} \left(\frac{\sigma_{n-1}(p-1)}{sp-n}\right)^{\frac{1}{p'}} r^{s-\frac{n}{p}} \|D^s_\delta u\|_{L^p(\mathbb{R}^n)},$$

since, as $n+(s-n)p' = \frac{sp-n}{p-1} > 0$,

$$\left(\int_{B(x,2r)} |x-z|^{(s-n)p'} dz\right)^{\frac{1}{p'}} = \left(\frac{\sigma_{n-1}(p-1)}{sp-n}\right)^{\frac{1}{p'}} (2r)^{s-\frac{n}{p}}$$
$$\leq 2^{1-\frac{n}{p}} \left(\frac{\sigma_{n-1}(p-1)}{sp-n}\right)^{\frac{1}{p'}} r^{s-\frac{n}{p}}.$$

Now, with respect to the second term, since $B(x,2r) \subset B(y,3r)$, we have

$$\begin{aligned}\int_{B(x,2r)} |y-z|^{s-n} |D^s_\delta u(z)| dz &\leq \int_{B(y,3r)} |y-z|^{s-n} |D^s_\delta u(z)| dz \\ &\leq \left(\int_{B(y,3r)} |y-z|^{(s-n)p'}\right)^{\frac{1}{p'}} dy \, \|D^s_\delta u\|_{L^p(\mathbb{R}^n)} \\ &\leq 3^{1-\frac{n}{p}} \left(\frac{\sigma_{n-1}(p-1)}{sp-n}\right)^{\frac{1}{p'}} r^{s-\frac{n}{p}} \|D^s_\delta u\|_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{6.59}$$

Finally, so as to tackle the last term, by the fundamental theorem of Calculus,

$$\begin{aligned}&\left|\frac{x-z}{|x-z|^{n+1-s}} - \frac{y-z}{|y-z|^{n+1-s}}\right| = \left|\int_0^1 \frac{d}{dt} \frac{tx+(1-t)y-z}{|tx+(1-t)y-z|^{n+1-s}} dt\right| \\ &= \Bigg| \int_0^1 (n+1-s) \frac{[tx+(1-t)y-z][(tx+(1-t)y-z)\cdot(x-y)]}{|tx+(1-t)y-z|^{n+3-s}} \\ &\qquad\qquad - \frac{x-y}{|tx+(1-t)y-z|^{n+1-s}} dt \Bigg| \\ &\leq \int_0^1 \left[(n+1-s)\frac{r}{|tx+(1-t)y-z|^{n+1-s}} + \frac{r}{|tx+(1-t)y-z|^{n+1-s}}\right] dt \\ &= (n+2-s) r \int_0^1 \frac{1}{|tx+(1-t)y-z|^{n+1-s}} dt,\end{aligned}$$

so

$$\int_{B(x,2r)^c} \left| \frac{x-z}{|x-z|^{n+1-s}} - \frac{y-z}{|y-z|^{n+1-s}} \right| |D^s_\delta u(z)| \, dz \leq$$
$$(n+2-s) r \int_0^1 \int_{B(x,2r)^c} |tx+(1-t)y-z|^{s-n-1} |D^s_\delta u(z)| \, dz \, dt.$$

By Hölder's inequality,

$$\int_{B(x,2r)^c} |tx+(1-t)y-z|^{s-n-1} |D^s_\delta u(z)| \, dz$$
$$\leq \left(\int_{B(x,2r)^c} |tx+(1-t)y-z|^{(s-n-1)p'} \, dz \right)^{\frac{1}{p'}} \|D^s_\delta u\|_{L^p(\mathbb{R}^n)} .$$

Since $B(tx+(1-t)y, r) \subset B(x, 2r)$ for all $t \in [0,1]$, we have

$$\int_{B(x,2r)^c} |tx+(1-t)y-z|^{(s-n-1)p'} \, dy \leq$$
$$\int_{B(tx+(1-t)y,r)^c} |tx+(1-t)y-z|^{(s-n-1)p'} \, dz =$$
$$\frac{\sigma_{n-1}}{(n+1-s)p'-n} r^{n+(s-n-1)p'},$$

since $n + (s-n-1)p' = -\frac{(1-s)p+n}{p-1} < 0$. Putting together the last three inequalities, we can see that there exists $\tilde{C} = \tilde{C}(s,n,p)$ such that

$$\begin{aligned} &\int_{B(x,2r)^c} \left| \frac{x-z}{|x-z|^{n+1-s}} - \frac{y-z}{|y-z|^{n+1-s}} \right| |D^s_\delta u(z)| \, dz \leq \\ &\tilde{C} r^{[n+(s-n-1)p'] \frac{1}{p'}+1} \|D^s_\delta u\|_{L^p(\mathbb{R}^n)} = \tilde{C} r^{s-\frac{n}{p}} \|D^s_\delta u\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{6.60}$$

Then, the conclusion follows combining (6.56), (6.57), (6.58), (6.59) and (6.60), as well as the inclusion $\operatorname{supp} D^s_\delta u \subset \Omega$, which implies $\|D^s_\delta u\|_{L^p(\mathbb{R}^n)} = \|D^s_\delta u\|_{L^p(\Omega)}$. □

Next we show a kind of 'nonlocal mean value theorem'. It is similar to the one based on the fractional gradient, Proposition 3.5.13. This is a key ingredient in order to show below compactness of embeddings in L^q spaces through the Fréchet–Kolmogorov theorem.

Proposition 6.4.4. *Let $M > 0$, $\delta > 0$, and $1 \leq p < \infty$. Then there exists $C > 0$ such that for all $s \in (0,1)$, $h \in B(0,M)$ and $u \in H^{s,p,\delta}_0(\Omega_{-\delta})$,*

$$\int_\Omega |u(x+h) - u(x)|^p dx \leq \left(\frac{C|h|^s}{s(1-s)} \right)^p \|D^s_\delta u\|^p_{L^p(\Omega)}.$$

Proof. By a standard density argument, it is enough to prove the result for $u \in C_c^\infty(\Omega)$. Let us fix $h \in \mathbb{R}^n$. By Theorem 6.3.6,

$$\begin{aligned} |u(x+h)-u(x)| &= \left| \int_{\mathbb{R}^n} (V_\delta^s(z) - V_\delta^s(z+h)) \cdot D_\delta^s u(x-z) dz \right| \\ &\le \int_{\mathbb{R}^n} |V_\delta^s(z) - V_\delta^s(z+h)| \, |D_\delta^s u(x-z)| dz. \end{aligned} \tag{6.61}$$

Notice that since $\operatorname{supp} u \subset \Omega_{-\delta}$, we have $\operatorname{supp} D_\delta^s u \subset \Omega$. Thus for every $z \in \Omega$ we have that $\operatorname{supp}(D_\delta^s u(\cdot + z)) \subset \Omega - z \subset \Omega - \Omega$. Let us take then $R > 0$ big enough such that $\Omega - \Omega \subset B(0,R)$.

By Theorem 6.3.14, there exists $C > 0$ such that

$$|V_\delta^s(z) - V_\delta^s(z+h)| \le C \left| \frac{z}{|z|^{n+1-s}} - \frac{z+h}{|z+h|^{n+1-s}} \right|,$$

for all $z \in B(0,R)$. Appling Hölder's inequality to the right hand side in (6.61),

$$\begin{aligned} |u(x+h)-u(x)| &\le C \left(\int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z+h}{|z+h|^{n+1-s}} \right| |D_\delta^s u(x-z)|^p dz \right)^{\frac{1}{p}} \\ &\qquad \left(\int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z+h}{|z+h|^{n+1-s}} \right| dz \right)^{\frac{1}{p'}} \le \\ &\left(\frac{C|h|^s}{s(1-s)} \right)^{\frac{1}{p'}} \left(\int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z+h}{|z+h|^{n+1-s}} \right| |D_\delta^s u(x-z)|^p dz \right)^{\frac{1}{p}} \end{aligned}$$

where we have used Lemma 3.5.12. Next, we integrate and apply Fubini's theorem to obtain

$$\begin{aligned} &\int_\Omega |u(x+h)-u(x)|^p \, dx \le \\ &\left(\frac{C|h|^s}{s(1-s)} \right)^{p/p'} \int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z+h}{|z+h|^{n+1-s}} \right| \int_\Omega |D_\delta^s u(x-z)|^p \, dx dz \le \\ &\left(\frac{C|h|^s}{s(1-s)} \right)^{p/p'+1} \|D_\delta^s u\|^p_{L^p(\mathbb{R}^n)} = \left(\frac{C|h|^s}{s(1-s)} \right)^{p} \|D_\delta^s u\|^p_{L^p(\Omega)}. \end{aligned}$$

□

Next, we write an analogous of Rellich–Kondrachov theorem for the space $H_0^{s,p,\delta}(\Omega_{-\delta})$. The compact embedding result is the following.

Theorem 6.4.5. *Set $0 < s < 1$, $0 < \delta$ and $1 < p < \infty$. Let $g \in H^{s,p,\delta}(\Omega)$. Then, for any sequence $\{u_j\}_{j\in\mathbb{N}} \subset H_g^{s,p,\delta}(\Omega_{-\delta})$ such that*

$$u_j \rightharpoonup u \quad \text{in } H^{s,p,\delta}(\Omega),$$

for some $u \in H^{s,p,\delta}(\Omega)$, one has $u \in H_g^{s,p,\delta}(\Omega_{-\delta})$ and

$$u_j \to u \quad \text{in } L^q(\Omega),$$

for every q satisfying

$$\begin{cases} q \in [1, p_s^*) & \text{if } sp < n, \\ q \in [1, \infty) & \text{if } sp = n, \\ q \in [1, \infty] & \text{if } sp > n. \end{cases}$$

where $p_s^* := \frac{np}{n-sp}$.

Proof. Clearly, $u \in H_g^{s,p,\delta}(\Omega_{-\delta})$, since $H_g^{s,p,\delta}(\Omega_{-\delta})$ is a closed affine subspace of $H^{s,p,\delta}(\Omega_{-\delta})$.

The case $sp > n$ follows from Theorem 6.4.3 and the Ascoli–Arzelà theorem. The case $sp = n$ reduces to the case $sp < n$. Thus, we focus on the case $sp < n$. Moreover, the case $q < p$ reduces to the case $q \geq p$, so we can assume that $q \in [p, p_s^*)$.

There exists $M > 0$ such that $\|u_j\|_{H^{s,p,\delta}(\Omega)} < M$ for each $j \in \mathbb{N}$. By Proposition 3.5.13 we have that for $j \in \mathbb{N}$

$$\|\tau_h u_j - u_j\|_{L^p(\Omega)} \leq \frac{C|h|^s}{s(1-s)} \|D_\delta^s u_j\|_{L^p(\Omega)}, \tag{6.62}$$

with $\tau_h u_j = u_j(\cdot - h)$. Next, as $p \leq q < p*$, we can write

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p*} \qquad \text{for some } \alpha \in (0, 1].$$

Let $C > 0$ denote a constant whose value may vary through the different steps. Finally, using the interpolation inequality, (6.62), the triangular inequality and Theorem 6.4.1,

$$\begin{aligned} \|\tau_h u_j - u_j\|_{L^q(\Omega)} &\leq \|\tau_h u_j - u_j\|_{L^p(\Omega)}^{\alpha} \|\tau_h u_j - u_j\|_{L^{p*}(\Omega)}^{1-\alpha} \\ &\leq \left(\frac{C|h|^s}{s(1-s)}\right)^{\alpha} \|D_\delta^s u_j\|_{L^p(\Omega)}^{\alpha} \left(2\|u_j\|_{L^{p*}(\Omega)}\right)^{1-\alpha} \\ &\leq \left(\frac{C|h|^s}{s(1-s)}\right)^{\alpha} \|D_\delta^s u_j\|_{L^p(\Omega)} \leq M \left(\frac{C|h|^s}{s(1-s)}\right)^{\alpha}. \end{aligned}$$

Thus,

$$\lim_{h\to 0}\sup_{j\in\mathbb{N}}\|\tau_h u_j - u_j\|_{L^q(\Omega)} = 0.$$

As a result, the Fréchet–Kolmogorov criterion leads to the compactness of $\{u_j\}_{j\in\mathbb{N}}$ in $L^q(\Omega)$, so finishing the proof. □

6.5 Comments on the fractional and nonlocal gradients

In this chapter we have already seen several results that allow us to make a comparison between the notions of fractional gradient and the nonlocal one. Thus, we can make some comments and observations here regarding the advantages and disadvantages of considering one operator or the other. As it was already mentioned at the introduction of Part II, [102] showed that the s-fractional gradient is the only fractional derivative up to a multiplicative constant verifying several natural requirements (invariance under translations and rotations, homogeneity under dilations and some continuity properties in an appropriate functional space). As for the nonlocal gradient, it seems it would be able to verify all of them except one, the homogeneity under dilations, in favour of being defined over bounded domains.

The fractional gradient seems to be the clear academic option when one wants to extend the concepts of differentiability to real exponents between 0 and 1. However, there are some properties that one would initially have thought to hold, but then it turned out it was not the case. For example, it is widely known that any (classical) derivative of a function $f \in \mathcal{S}$ (the Schwartz space) remains in such space, however it does not hold that $D^s(\mathcal{S}) \subset \mathcal{S}$. This assertion can easily be seen in the formula of the Fourier transform of the fractional gradient, which exhibits some differentiability problems at $\xi = 0$, preventing it from being a Schwartz function. Actually, the same issue might be observed in the set of (tempered) distributions, a set that was conceived as a set where one could derivate (with a natural exponent) indefinitely. Sometimes, the fractional gradient of a distribution cannot even be defined (it can for distributions with compact supports or L^p functions seen as distributions). This issue is not exhibited by the nonlocal gradient D^s_δ, since $D^s_\delta(C^\infty_c(\mathbb{R}^n)) \subset C^\infty_c$ and $D^s_\delta(\mathcal{S}) \subset \mathcal{S}$ (see Proposition 6.3.3). Hence, by duality, the corresponding distributional spaces are also close under this operation.

Therefore, besides the common properties of both notions, typical characteristics of the *fractional gradient* include being defined over the whole space and homogeneity under dilations. It also enjoys a semi-group property and it is closely linked to Bessel fractional spaces and the fractional laplacian as we have seen in Chapter 3. According to the formulas shown in Part II and the clear resemblance to they analogous ones in the local case, the fractional gradient might be seen as the suitable generalization from an academic approach.

As for the *nonlocal gradient*, it is defined over bounded domains (which is relevant for applications), and thus, there could be considered more general nonlocal boundary conditions. Nevertheless, it does not seem to enjoy a semi-group property. As opposed to the fractional gradient, we have that given E a space of test functions, in particular $E = \mathcal{S}$ or $E = C_c^\infty(\mathbb{R}^n)$, then $D_\delta^s(E) \subset E$. By duality, such property is inherited by the corresponding distributional spaces. Finally, as with the fractional gradient, a nonlocal laplacian can be defined from the nonlocal gradient. This is shown in the next section.

6.5.1 Nonlocal laplacian

Given the definitions of nonlocal gradient and divergence, it is natural to consider the composition of those two in order to obtain a sort of nonlocal laplacian. Actually, such issue is addressed in [44] where Definition 6.1.1 would be a particular case of what they call weighted nonlocal gradient and divergence. Here we would like to see that the operator obtained with this particular kernel ρ_δ (the Riesz potential times a cut-off function) gives rise to a notion where several equivalent characterization can be given. This is something that does not hold with the settled definitions for a fractional laplacian over bounded domain (as opposed to its counterpart over the whole domain). In fact, for $0 < s < 1$, $0 < \delta$ and $u \in C_c^\infty(\mathbb{R}^n)$, we have that the following characterizations are equivalent. We recall that the function Q_δ^s was defined in Lemma 6.3.2.

1) Fourier transform:

$$\widehat{\Delta_\delta^s u}(\xi) = -4\pi^2|\xi|^2\widehat{Q_\delta^s}^2(\xi)\hat{u}(\xi).$$

2) Nonlocal divergence of the nonlocal gradient:

$$\Delta_\delta^s u(x) = \operatorname{div}_\delta^s D_\delta^s u(x).$$

3) Inverse of a potencial

$$\Delta_\delta^s u(x) = (n+s-1)^2 \int_{B(x,2\delta)} u(y)\beta(x-y)\,dy$$

where $\hat{\beta} = (\widehat{\nabla Q_\delta^s * \nabla} Q_\delta^s)$

$$\beta(\tilde{x}) = \int \frac{z}{|z|}\rho_\delta(z)\frac{z-\tilde{x}}{|z-\tilde{x}|}\rho_\delta(z-\tilde{x})\,dz.$$

We start with the definition from 2). Let $u \in C_c^\infty(\mathbb{R}^n)$, if we take into account (6.11), (6.14) and (6.15) we have that

$$\Delta_\delta^s u(x) = \operatorname{div}_\delta^s D_\delta^s u(x) = \sum_{i=1}^n D_{\delta,i}^s(D_{\delta,i}^s u)(x).$$

Applying Corollary 6.3.4 twice it yields

$$\widehat{\Delta_\delta^s} u(\xi) = \sum_{i=1}^n 2\pi i\xi_i\,\widehat{D_{\delta,i}^s u}(\xi)\,\hat{Q}_\delta^s(\xi) = -\sum_{i=1}^n 4\pi^2\xi_i^2\hat{u}\hat{Q}_\delta^s(\xi)^2 = -4\pi^2|\xi|^2\widehat{Q_\delta^s * Q}_\delta^s\hat{u},$$

where $\hat{Q}_\delta^s(\xi)^2 = \widehat{Q_\delta^s * Q_\delta^s}$ by the Fourier transform of the convolution of two integrable functions.

Finally, a step forwards in the previous equation (i.e. $\hat{\nabla} f(\xi) = 2\pi i\xi\hat{f}(\xi)$) gives us that

$$\widehat{\Delta_\delta^s} u(\xi) = (\widehat{\nabla Q_\delta^s * \nabla} Q_\delta^s)(\xi)\hat{u}(\xi),$$

then, taking inverse Fourier transform and recalling Lemma 6.3.2 so as to define β, 3 follows.

Chapter 7. Existence of minimizers of nonlocal energy functionals. Euler-Lagrange equations

Finally, in a similar way to what was done in Part II, the results collected in the previous chapter allow us to study the existence of minimizers of nonlocal functionals. First, we consider the scalar case under hypothesis of convexity. Then, we proceed with the computation of the Euler-Lagrange equations as the equilibrium equations of the system. Most part of this chapter is devoted to the analysis of vector variational problems involving the nonlocal gradient D^s_δ. Specifically, we want to study the existence of minimizers of polyconvex energy functionals based on D^s_δ (see Definition 4.0.1 for the notion of polyconvexity). The corresponding Euler-Lagrange will also be shown. This time, as opposed to the one addressed in Chapter 4, the energy functional would be defined over bounded domains, making it suitable as a nonlocal model for hyperelasticity. At the moment, we are going to consider just Dirichlet (nonlocal) boundary conditions on the tubular neighbourhood $\Omega_\delta \backslash \Omega_{-\delta}$, which, although the radius (2δ) is actually imposed by the embedding theorem from the previous chapter, make sense since we would be taking two nonlocal derivatives of a function in the model. Actually, as happened in the fractional case, we will have to deal with a nonlocal Piola identity

$$\operatorname{Div}^s_\delta \operatorname{cof} D^s_\delta u = 0$$

(where $\operatorname{Div}^s_\delta$ means the nonlocal divergence by rows). This is a nonlocal version of that of Theorem 4.1.2 and, as it was appointed at the introduction of Chapter 4, it might be useful in other contexts. In fact, we refer to the introduction of such chapter, since the steps followed here so as to obtain the existence of minimizers are practically the same. We recall that although we have shown the symmetry of second (nonlocal) derivatives (Proposition 6.2.6), it is not enough for the proof of a nonlocal Piola identity, since, as happened in the fractional case, the nonlocal version of the Leibniz formula gives rise to non symmetric terms, making the proof of such identity more difficult. Either way, once we had reached a particular step, we would be able to refer ourselves directly to the fractional case, so as to conclude the proof of the nonlocal Piola identity.

7.1 Convex functionals

We start this chapter with we prove the existence of minimizers of functionals of the form

$$I(u) = \int_\Omega W(x, u(x), D^s_\delta u(x))\, dx \tag{7.1}$$

under coercivity and convexity conditions. We also show in the next section the corresponding Euler–Lagrange equations satisfied by the minimizers.

The result on the existence of minimizers, which is a standard application of the direct method of the Calculus of Variations, is as follows.

Theorem 7.1.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $p > 1$ and $0 < s < 1$. Let $u_0 \in H^{s,p,\delta}(\Omega)$. Let $W : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ satisfy the following conditions:*

a) W is $\mathcal{L}^n \times \mathcal{B} \times \mathcal{B}^n$-measurable, where $\mathcal{L}^n$ denotes the Lebesgue sigma-algebra in $\mathbb{R}^n$, whereas $\mathcal{B}$ and $\mathcal{B}^n$ denote the Borel sigma-algebras in $\mathbb{R}$ and $\mathbb{R}^n$, respectively.

b) $W(x, \cdot, \cdot)$ is lower semicontinuous for a.e. $x \in \mathbb{R}^n$.

c) For a.e. $x \in \Omega$ and every $y \in \mathbb{R}$, the function $W(x, y, \cdot)$ is convex.

d) There exist $c > 0$ and $a \in L^1(\Omega)$ such that

$$W(x, y, F) \geq a(x) + c\,|F|^p$$

for a.e. $x \in \Omega$, all $y \in \mathbb{R}$ and all $F \in \mathbb{R}^n$.

Define I as in (7.1), and assume that I is not identically infinity in $H^{s,p,\delta}_{u_0}(\Omega_{-\delta})$. Then there exists a minimizer of I in $H^{s,p,\delta}_{u_0}(\Omega_{-\delta})$.

Proof. Assumption *d)* shows that the functional I is bounded below by $\int a$. As I is not identically infinity in $H^{s,p,\delta}_{u_0}(\Omega_{-\delta})$, there exists a minimizing sequence $\{u_j\}_{j\in\mathbb{N}}$ of I in $H^{s,p,\delta}_{u_0}(\Omega_{-\delta})$. Assumption *d)* implies that $\{D^s_\delta u_j\}_{j\in\mathbb{N}}$

is bounded in $L^p(\Omega, \mathbb{R}^n)$. By Theorem 6.4.1, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega)$. Therefore, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $H^{s,p,\delta}(\Omega)$. As $H^{s,p,\delta}(\Omega)$ is reflexive (Proposition 6.1.4), we can extract a weakly convergent subsequence. Using Theorem 6.4.5, we obtain that there exists $u \in H^{s,p,\delta}(\Omega)$ such that for a subsequence (not relabelled),

$$u_j \rightharpoonup u \text{ in } H^{s,p,\delta}(\Omega) \quad \text{and} \quad u_j \to u \text{ in } L^p(\Omega). \tag{7.2}$$

Moreover, $u \in H^{s,p,\delta}_{u_0}(\Omega_{-\delta})$.

A standard lower semicontinuity result for convex functionals (see, e.g., [59, Th. 7.5]) shows that

$$I(u) \leq \liminf_{j\to\infty} I(u_j).$$

Therefore, u is a minimizer of I in $H^{s,p,\delta}_{u_0}(\Omega_{-\delta})$ and the proof is concluded. □

7.2 Euler-Lagrange equations

We show in this section the Euler–Lagrange equation satisfied by any minimizer.

Theorem 7.2.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $p > 1$ and $0 < s < 1$. Let $u_0 \in H^{s,p,\delta}(\Omega)$. Let $W : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy the following conditions:*

a) $W(\cdot, y, F)$ is $\mathcal{L}^n$-measurable for each $y \in \mathbb{R}$ and $F \in \mathbb{R}^n$, where $\mathcal{L}^n$ denotes the Lebesgue sigma-algebra in $\mathbb{R}^n$.

b) $W(x, \cdot, \cdot)$ is of class C^1 for a.e. $x \in \Omega$.

c) There exist $c > 0$, $q \in [1, \infty)$, $a \in L^1(\Omega)$ and a function $f : \mathbb{R} \to \mathbb{R}$ sending bounded sets in bounded sets such that

$$|W(x,y,F)| + |D_y W(x,y,F)| + |D_F W(x,y,F)| \leq \begin{cases} a(x) + c\left(|y|^{p^*} + |F|^p\right) & \text{if } sp < n, \\ a(x) + c\left(|y|^q + |F|^p\right) & \text{if } sp = n, \\ a(x) + f(y) + c\,|F|^p & \text{if } sp > n, \end{cases}$$

for a.e. $x \in \Omega$, all $y \in \mathbb{R}$ and all $F \in \mathbb{R}^n$.

Define I as in (4.34). Let u be a minimizer of I in $H^{s,p,\delta}_{u_0}(\Omega_{-\delta})$. Then, for every $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \left[D_y W(x, u(x), D^s_\delta u(x))\, \varphi(x) + D_z W(x, u(x), D^s_\delta u(x)) \cdot D^s_\delta \varphi(x) \right] dx = 0. \tag{7.3}$$

If, in addition, $D_z W(\cdot, u(\cdot), D^s_\delta u(\cdot)) \in C^1(\Omega_\delta, \mathbb{R}^n)$ then

$$\begin{aligned} &D_y W(x, u(x), D^s_\delta u(x)) = \\ &\operatorname{div}^s_\delta D_z W(x, u(x), D^s_\delta u(x)) + \\ &(n+s-1) \int_{\Omega_{B,\delta}} \frac{D_z W(y, u_0(y), D^s_\delta u(y))}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y)\, dy \end{aligned} \tag{7.4}$$

for a.e. $x \in \Omega_{-\delta}$.

Proof. Using a standard argument, in order to show (7.3) it is enough to check that one can differentiate under the integral sign in the function $t \mapsto I(u+t\varphi)$. Assumption *c)* shows that this is the case (see, e.g., [74, Ch. 13, §2, Lemma 2.2]). Indeed, in the case $sp < n$ we use Theorem 6.4.1; in the case $sp = n$ we use Theorem 6.4.1 and the embedding $H^{s,\frac{n}{s},\delta}(\Omega) \subset H^{s,q,\delta}(\Omega)$ for all $q < \frac{n}{s}$. In the case $sp > n$ we use the embedding provided by Theorem 6.4.3. Thus, (7.3) is proved.

In order to derive (7.4) from (7.3) we make the abbreviation $D_z W(x)$ for $D_z W(x, u(x), D^s_\delta u(x))$. We use Theorem 6.1.2 to obtain

$$\begin{aligned} &\int_\Omega D_z W(x) \cdot D^s_\delta \varphi(x)\, dx = \\ &- \int_\Omega \varphi(x) \operatorname{div}^s_\delta DW(x)\, dx \\ &-(n+s-1) \int_\Omega \int_{\Omega_{B,\delta}} \frac{\varphi(x)\, DW(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y)\, dy dx. \end{aligned}$$

Then we combine it with (7.3), apply the fundamental lemma of the Calculus of Variations and (7.4) follows. □

7.3 Polyconvex functionals

7.3.1 Nonlocal Piola Identity

In this section we introduce a nonlocal version of the Piola Identity. In the classical case, the Piola identity can be easily computed thanks to the symmetry of the second derivatives (Schwartz Theorem), however, as it was aforementioned, the nonlocal version of the Leibniz rule (derivative of a product) does not mimic exactly its local counterpart, which causes the appearance of non symmetric terms. Therefore, although Proposition 6.2.6 may be enough to prove the nonlocal Piola identity when $n = 2$, it is not so for $n \geq 3$, when nonlinearities appear in such computation.

This identity is the main step in order to prove the existence of solutions to our vectorial nonlocal energy, since it will allow us to prove the weak continuity in $H^{s,p,\delta}$ of the determinant of the nonlocal gradient. We recall that cof, the cofactor matrix, satisfies $\operatorname{cof} A\, A^T = (\det A)\, I$ for every $A \in \mathbb{R}^{n\times n}$.

In this section we will extensively employ the following formulas for the nonlocal gradient and divergence, obtained from Definition 6.1.1 through odd symmetry.

$$\begin{aligned} D_\delta^s u &= -\operatorname{pv}_x n \int_{B(x,\delta)} \frac{u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{w_\delta(|x-y|)}{|x-y|^{n+s-1}}\, dy, \\ \operatorname{div}_\delta^s \phi(x) &= -\operatorname{pv}_x n \int_{B(x,\delta)} \frac{\phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \frac{w_\delta(|x-y|)}{|x-y|^{n+s-1}}\, dy \end{aligned} \tag{7.5}$$

for $x \in \Omega$, $u : \Omega_\delta \to \mathbb{R}$ and $\phi : \Omega_\delta \to \mathbb{R}^n$.

Given the strong similarity with the fractional Piola identity (subsection 4.1) we refer to the introduction of such section as a foretaste of the ideas and steps involved. Moreover, in this and the next sections we will employ again the notation for the submatrices shown in Definition 4.1.1.

The following lemma will be useful in the proof of the nonlocal Piola identity.

Lemma 7.3.1. *Let $k \in \mathbb{N}$ be with $1 \leq k \leq n$. Consider indices $1 \leq j_1 < \cdots < j_k \leq n$ and let $N = N_{j_1,\ldots,j_k}$ be the function of Definition 4.1.1. Then there exists a continuous function $G : [0,\infty) \times (\mathbb{R}^n)^{k-1} \to \mathbb{R}$ such that for any $a_1, \ldots, a_k \in \mathbb{R}^n$ and $b_0\delta > \epsilon_1, \ldots, \epsilon_k > 0$ we have*

$$\left| \int_{\left(\bigcup_{j=1}^k B(a_j,\epsilon_j)\right)^c} \frac{\det([x-a_1]_N, \ldots, [x-a_k]_N)}{|x-a_1|^{n+s+1} \cdots |x-a_k|^{n+s+1}} w_\delta(x-a_1) \ldots w_\delta(x-a_n) dx \right| \leq \frac{\epsilon_1^{1-s}}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} G(\epsilon_1, a_2 - a_1, \ldots, a_k - a_1),$$

where b_0 is the constant from the definition of w_δ.

Proof. We can assume that the points $a_1, \ldots, a_k$ do not lie on an affine manifold of dimension $k-2$, since otherwise $\det([x-a_1]_N, \ldots, [x-a_k]_N) = 0$ for all $x \in \mathbb{R}^n$.

Now, let Q^s_δ be the function from Lemma 6.3.2, then $Q^s_\delta \in C^\infty(\mathbb{R}^n\backslash\{0\}) \cap L^1(\mathbb{R}^n)$, $Q^s_\delta = 0$ in $B(0,\delta)^c$ and

$$\nabla Q^s_\delta(x) = -c_{n,s}\frac{x}{|x|^{n+s+1}}w_\delta(x),$$

and by Lemma 6.3.2 again, there exists $z_0 \in \mathbb{R}$ such that

$$Q^s_\delta(x) = \frac{a_0}{\gamma(1-s)|x|^{n+s-1}} + \frac{z_0}{\gamma_{(1-s)}} \quad \forall x \in B(0, b_0\delta) \tag{7.6}$$

where b_0 is the constant from the definition of w_δ. Next define $h_i : \mathbb{R}^n \setminus \{a_i\} \to \mathbb{R}$ as $h_i(x) = \frac{-1}{c_{n,s}}Q^s_\delta(x-a_i)$, for each $i = 1,\dots,k$. Define $H_\delta : \mathbb{R}^n \setminus \{a_1,\dots,a_k\} \to \mathbb{R}^k$ componentwise as $H_\delta = (h_1,\dots,h_k)^T$. Then

$$DH(x) = \begin{pmatrix} \nabla h_1(x) \\ \vdots \\ \nabla h_k(x) \end{pmatrix} = \begin{pmatrix} \frac{(x-a_1)w_\delta(x-a_1)}{|x-a_1|^{n+s+1}} \\ \vdots \\ \frac{(x-a_k)w_\delta(x-a_k)}{|x-a_k|^{n+s+1}} \end{pmatrix}. \tag{7.7}$$

Call $\vec{\jmath} = (j_1,\dots,j_k)$ and denote by $D_{\vec{\jmath}}H_\delta$ the submatrix of DH_δ formed by the columns $j_1,\dots,j_k$. Then, for all $x \in \mathbb{R}^n \setminus \{a_1,\dots,a_k\}$,

$$\det D_{\vec{\jmath}}H_\delta(x) = \frac{\det\Big([x-a_1]_N w_\delta(x-a_1),\dots,[x-a_k]_N w_\delta(x-a_k)\Big)}{|x-a_1|^{n+s+1}\cdots|x-a_k|^{n+s+1}}. \tag{7.8}$$

Let $R > 0$ be big enough so that $\bigcup_{j=1}^k \bar{B}(a_j,\delta) \subset B(0,R)$, then $\operatorname{supp} H_\delta \subset B(0,R)$, as a result

$$\int_{\left(\bigcup_{j=1}^k B(a_j,\epsilon_j)\right)^c} \det D_{\vec{\jmath}}H_\delta = \int_{B(0,R)\backslash\bigcup_{j=1}^k B(a_j,\epsilon_j)} \det D_{\vec{\jmath}}H_\delta.$$

As H_δ is smooth outside $\bigcup_{j=1}^k B(a_j,\epsilon_j)$, we have that

$$\det D_{\vec{\jmath}}H_\delta = \operatorname{div}[h_1(\operatorname{cof} D_{\vec{\jmath}}H_\delta)_1]_{\bar{N}},$$

where $(\operatorname{cof} D_{\vec{\jmath}}H_\delta)_1$ indicates the first row of $\operatorname{cof} D_{\vec{\jmath}}H_\delta$, and $[\cdot]_{\bar{N}} = [\cdot]_{\bar{N}_{j_1,\dots,j_k}}$ is the function from Definition 4.1.1. By the divergence theorem,

$$\int_{B(0,R)\backslash\bigcup_{j=1}^k B(a_j,\epsilon_j)} \det D_{\vec{\jmath}}H_\delta = -\int_{\partial\bigcup_{j=1}^k B(a_j,\epsilon_j)} [h_1(\operatorname{cof} D_{\vec{\jmath}}H_\delta)_1]_{\bar{N}} \cdot \nu_j, \tag{7.9}$$

where $\nu_j(x) = \frac{x-a_j}{\epsilon_j}$ in $\partial B(a_j,\epsilon_j)$ for $j = 1,\dots,k$, and we have used that $h_1(x) = 0$ in $\partial B(0,R)$ (recall that $h_1 = 0$ in $B(0,a_1)^c \supset B(0,R)^c$).

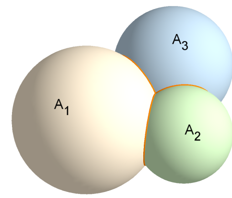


Figure 7.1: Sets A_1, A_2, A_3 in $\mathbb{R}^3$

We continue as in the proof of Lemma 4.1.1. For each $i = 1, \dots, n$ we set

$$A_i = \partial\left(\bigcup_{j=1}^{k} B(a_j, \epsilon_j)\right) \cap \partial B(a_i, \epsilon_i).$$

As a consequence of the inclusion $\partial \bigcup_{j=1}^{k} B(a_j, \epsilon_j) \subset \bigcup_{j=1}^{k} \partial B(a_j, \epsilon_j)$, we have that

$$\partial \bigcup_{j=1}^{k} B(a_j, \epsilon_j) = \bigcup_{j=1}^{k} A_j.$$

Moreover, the $(n-1)$-dimensional area of $A_i \cap A_j$ is zero for $1 \le i < j \le k$. Figure 7.1 illustrates this situation when $k = n = 3$.

Next, using (4.4) and (7.7), we have that for $j = 2, \dots, k$ and $x \in \partial B(a_j, \epsilon_j)$,

$$[h_1(\operatorname{cof} D_{\bar{j}}H)_1]_{\bar{N}} \cdot \nu_j(x) = \frac{\det\Big([x-a_j]_N, [x-a_2]_N w_\delta(x-a_2), \dots, [x-a_k]_N w_\delta(x-a_k)\Big)}{|x-a_j||x-a_2|^{n+s+1} \cdots |x-a_k|^{n+s+1}} = 0.$$

As a result, recalling (7.9) and the inclusion $A_j \subset \partial B(a_j, \epsilon_j)$, we have that for every $x \in A_1$,

$$\int_{\left(\bigcup_{j=1}^{k} B(a_j,\epsilon_j)\right)^c} \det D_{\bar{j}}H \, dx = -\int_{A_1} [h_1(\operatorname{cof} D_{\bar{j}}H)_1]_{\bar{N}} \cdot \nu_1 \, dS. \tag{7.10}$$

Next, if we denote $\bar{h} = \frac{-1}{(n+s-1)|x|^{n+s-1}}$, $\bar{h}_1 = \bar{h}(x-a_1)$ and recall (7.6), we have that for every $x \in A_1 \subset \partial B(a_1, \epsilon_1)$ with $0 < \epsilon_1 < b_0\delta$, $\bar{h}_1(x) = -\frac{1}{(n+s-1)\epsilon_1^{n+s-1}}$ and

$$h_1(x) = a_0\bar{h}_1(x) + \frac{z_0}{\gamma_{(1-s)}} = a_0\bar{h}_1(x) - \frac{z_0}{\gamma_{(1-s)}}(n+s-1)\epsilon_1^{n+s-1}\bar{h}_1(x) = z(\epsilon_1)\bar{h}_1(x) \tag{7.11}$$

with $z(\epsilon_1) \in C^{n-1}(\mathbb{R})$ such that $z(0) = a_0$.

Having in mind the expression (7.11), the multilinearity of the determinant and considering (4.4) and (7.7), we have that, for $x \in A_1$,

$$
\begin{gathered}
-\left[h_1(\operatorname{cof} D_{\vec{j}}H)_1\right]_{\bar{N}} \cdot \nu_1(x) = \frac{z(\epsilon)}{n+s-1}\frac{1}{\epsilon_1^{n+s}}(\operatorname{cof} D_{\vec{j}}H)_1 \cdot [x-a_1]_N = \\
\frac{1}{n+s-1}\frac{z(\epsilon)}{\epsilon_1^{n+s}}\frac{\det([x-a_1]_N,[x-a_2]_N w_\delta(x-a_2),\dots,[x-a_k]_N w_\delta(x-a_k))}{|x-a_2|^{n+s+1}\cdots|x-a_k|^{n+s+1}} = \\
\frac{1}{n+s-1}\frac{z(\epsilon)}{\epsilon_1^{n+s}}\frac{\det([x-a_1]_N,[a_1-a_2]_N w_\delta(x-a_2),\dots,[a_1-a_k]_N w_\delta(x-a_k))}{|x-a_2|^{n+s+1}\cdots|x-a_k|^{n+s+1}} = \\
\frac{1}{n+s-1}\frac{z(\epsilon)}{\epsilon_1^{n+s-1}}\frac{([\operatorname{cof}([x-a_1]_N, Y^\delta_{a_2,\dots,a_n}(a_1,x))]_{\bar{M}})_1}{|x-a_2|^{n+s+1}\cdots|x-a_k|^{n+s+1}} \cdot \nu_1(x),
\end{gathered}
\tag{7.12}
$$

where

$$
Y^\delta_{a_2,\dots,a_n}(a_1,x) := \big([a_1-a_2]_N w_\delta(x-a_2),\dots,[a_1-a_k]_N w_\delta(x-a_k)\big)
$$

and $[\cdot]_{\bar{M}} = [\cdot]_{\bar{M}_{i_1,\dots,i_k;j_1,\dots,j_k}}$ is the function from Definition 4.1.1.

Let Π_k be the only hyperplane in $\mathbb{R}^k$ such that the points $[a_1]_N,\dots,[a_k]_N$ belong to Π_k, and consider one of the two unit normals $\vec{n} \in \mathbb{R}^k$ to Π_k. Let $T_k : \mathbb{R}^k \to \mathbb{R}^k$ be the symmetry with respect to Π_k, so that for every $y \in \mathbb{R}^k$,

$$
T_k y = y - 2(y - [a_1]_N)\cdot \vec{n}. \tag{7.13}
$$

Let $\vec{m} = [\vec{n}]_{\bar{N}}$, and let Π be the affine hyperplane in $\mathbb{R}^n$ with normal $\vec{m}$ passing through a_1. Consider $T : \mathbb{R}^n \to \mathbb{R}^n$ as the symmetry across Π. Then, for all $x \in \mathbb{R}^n$,

$$
Tx = x - 2(x-a_1)\cdot \vec{m}. \tag{7.14}
$$

Let $a_{k+1},\dots,a_n \in \Pi$ be such that the points $a_1,\dots,a_n$ do not lie in an affine manifold of dimension $n-2$. Define $A_1^\pm = \{x \in A_1 : \pm\det(x-a_1, a_1-a_2,\dots,a_1-a_n) > 0\}$. Then $T(A_1^\pm) = A_1^\mp$, and $A_1^+ \cup A_1^-$ cover A_1 up to a set of zero $(n-1)$-measure; see Figure 7.2. Using the change of variables formula (4.3), we obtain

$$
\begin{aligned}
&\int_{A_1^-}\frac{([\operatorname{cof}([x-a_1]_N, Y^\delta_{a_2,\dots,a_n}(a_1,x))]_{\bar{M}})_1}{|x-a_2|^{n+s+1}\cdots|x-a_k|^{n+s+1}} \cdot \nu_1(x)\,dS(x) \\
&= \int_{A_1^+}\frac{([\operatorname{cof}([Tx-a_1]_N, Y^\delta_{a_2,\dots,a_n}(a_1,Tx))]_{\bar{M}})_1}{|Tx-a_2|^{n+s+1}\cdots|Tx-a_k|^{n+s+1}} \cdot \nu_1(Tx)\,dS(x).
\end{aligned}
\tag{7.15}
$$

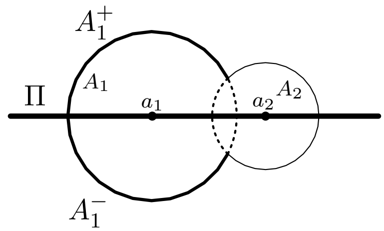


Figure 7.2: Sets A_1, A_2, A_1^+, A_1^- and Π

Now, thanks to (4.4), for $x \in A_1^+$,

$$\begin{aligned}&([\mathrm{cof}([Tx-a_1]_N, Y^\delta_{a_2,\ldots,a_n}(a_1,Tx))]_{\bar{M}})_1 \cdot \nu_1(Tx) = \\ &\frac{1}{\epsilon_1}\det([Tx-a_1]_N, Y^\delta_{a_2,\ldots,a_n}(a_1,Tx)).\end{aligned} \tag{7.16}$$

Let $\vec{T}_k : \mathbb{R}^k \to \mathbb{R}^k$ be the linear map corresponding to the affine map T_k, and, analogously, $\vec{T} : \mathbb{R}^n \to \mathbb{R}^n$ the linear map corresponding to T. We notice that $\det \vec{T}_k = -1$. Having in mind (7.13) and (7.14), we find that

$$\vec{T}_k y = y - 2y \cdot \vec{n}, \qquad y \in \mathbb{R}^k$$

and

$$\vec{T}x = x - 2x \cdot \vec{m}, \qquad x \in \mathbb{R}^n,$$

from which we deduce that $\vec{T}_k \circ [\cdot]_N = [\cdot]_N \circ \vec{T}$. Thus, by the multilinearity of the determinant we can write next equation without the scalar terms $(\frac{w_\delta(Tx-a_j)}{|Tx-a_j|^{n+s+1}})$, $j = 2, \ldots, k$.

$$\begin{aligned}&\det\left([Tx-a_1]_N, [a_1-a_2]_N, \ldots, [a_1-a_k]_N\right) \\ &= \det([Tx-Ta_1]_N, [Ta_1-Ta_2]_N, \ldots, [Ta_1-Ta_k]_N) \\ &= \det([\vec{T}(x-a_1)]_N, [\vec{T}(a_1-a_2)]_N, \ldots, [\vec{T}(a_1-a_k)]_N) \\ &= \det(\vec{T}_k([x-a_1]_N), \vec{T}_k([a_1-a_2]_N), \ldots, \vec{T}_k([a_1-a_k]_N)) \\ &= \det \vec{T}_k([x-a_1]_N, [a_1-a_2]_N, \ldots, [a_1-a_k]_N) \\ &= -\det([x-a_1]_N, [a_1-a_2]_N, \ldots, [a_1-a_k]_N).\end{aligned} \tag{7.17}$$

Putting together (7.15), (7.16) and (7.17), we obtain that

$$\int_{A_1^-} \frac{\det([x-a_1]_N, [a_1-a_2]_N w_\delta(x-a_2), \dots, [a_1-a_k]_N w_\delta(x-a_k))}{|x-a_2|^{n+s+1} \cdots |x-a_k|^{n+s+1}} \, dS(x) =$$
$$- \int_{A_1^+} \frac{\det([x-a_1]_N, [a_1-a_2]_N w_\delta(Tx-a_2), \dots, [a_1-a_k]_N w_\delta(Tx-a_k))}{|Tx-a_2|^{n+s+1} \cdots |Tx-a_k|^{n+s+1}} \, dS(x).$$

Consequently, when we define $f : \mathbb{R}^n \setminus \{a_2, \dots, a_k\} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ as

$$f(y) := \frac{1}{(|y-a_2| \cdots |y-a_k|)^{n+s+1}}, \quad g(y) := w_\delta(y-a_2) \dots w_\delta(y-a_k)$$

and $J : \mathbb{R}^n \setminus \{a_2, \dots, a_k\} \to \mathbb{R}$ as $J(y) = f(y)g(y)$, we have that

$$\begin{aligned} &\int_{A_1} \frac{\det([x-a_1]_N, [a_1-a_2]_N w_\delta(x-a_2), \dots, [a_1-a_k]_N w_\delta(x-a_k))}{|x-a_2|^{n+s+1} \cdots |x-a_k|^{n+s+1}} \, dS(x) = \\ &\int_{A_1^+} \det([x-a_1]_N, [a_1-a_2]_N, \dots, [a_1-a_k]_N) \, [J(x) - J(Tx)] \, dS(x). \end{aligned} \tag{7.18}$$

For every $x \in A_1^+$, we join x with Tx by a curve γ_x inside A_1, and note that the length of γ_x can be taken to be bounded by $2\pi\varepsilon_1$. Accordingly, let $\gamma_x : [0,1] \to A_1$ be of class C^1 such that $\gamma_x(0) = x$, $\gamma_x(1) = Tx$ and $|\gamma_x'|$ is constant with $|\gamma_x'| \leq 2\pi\varepsilon_1$. Then

$$\begin{aligned} |J(x) - J(Tx)| = |J(\gamma_x(0)) - J(\gamma_x(1))| &\leq \int_0^1 |\gamma_x'| \, |\nabla J(\gamma_x(t))| \, dt \\ &\leq 2\pi\epsilon_1 \int_0^1 |\nabla J(\gamma_x(t))| \, dt. \end{aligned} \tag{7.19}$$

By (4.18) in Lemma 4.1.1,

$$|\nabla f(y)| = (n+s+1) \left(|y-a_2| \cdots |y-a_k|\right)^{-n-s-2} \sum_{i=2}^{k} \prod_{\substack{j=2 \\ j \neq i}}^{k} |y-a_j| \, ,$$

for $y \in \mathbb{R}^n \setminus \{a_2, \dots, a_k\}$.

Again, by the same computation in Lemma 4.1.1, as $|y-a_j| > \epsilon_j$ for every

$y \in A_1$ and $j \in \{2, \ldots, k\}$,

$$|\nabla f(y)| \leq \frac{n+s+1}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} \sum_{i=2}^{k} \prod_{\substack{j=2 \\ j\neq i}}^{k} |y - a_j|$$
$$\leq \frac{n+s+1}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} \sum_{i=2}^{k} \prod_{\substack{j=2 \\ j\neq i}}^{k} (\epsilon_1 + |a_1 - a_j|).$$

Then, since g is smooth with compact support, we have that there exists $C > 0$ such that

$$|\nabla J(y)| \leq |\nabla f(y)|\,|g(y)| + |f(y)|\,|\nabla(y)|$$
$$\leq C \frac{n+s+1}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} \sum_{i=2}^{k} \prod_{\substack{j=2 \\ j\neq i}}^{k} (\epsilon_1 + |a_1 - a_j|) + \frac{C}{(\epsilon_2 \ldots \epsilon_k)^{n+s+1}}$$

so with (7.19) we obtain that

$$|J(x) - J(Tx)| \leq$$
$$2\pi\epsilon_1 C \left(\frac{n+s+1}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} \sum_{i=2}^{k} \prod_{\substack{j=2 \\ j\neq i}}^{k} (\epsilon_1 + |a_1 - a_j|) + \frac{1}{(\epsilon_2 \ldots \epsilon_k)^{n+s+1}} \right). \tag{7.20}$$

On the other hand, for all $x \in A_1$,

$$|\det([x - a_1]_N, [a_1 - a_2]_N, \ldots, [a_1 - a_k]_N)| \leq$$
$$k!\,|x - a_1| \prod_{j=2}^{k} |a_1 - a_j| = k!\,\epsilon_1 \prod_{j=2}^{k} |a_1 - a_j|\,. \tag{7.21}$$

Putting together (7.8), (7.10), (7.12), (7.18), (7.20), (7.21) and the function $z(\epsilon_1)$, as well as the fact that the $(n-1)$-dimensional area of A_1^+ is bounded by a constant times ϵ_1^{n-1} and that $0 < \epsilon_i < b_0\delta$, $i = 1, \ldots, n$, we obtain that, for a constant $C > 0$ depending on n, s and a_0,

$$\left| \int_{\left(\bigcup_{j=1}^{k} B(a_j,\epsilon_j)\right)^c} \frac{\det([x - a_1]_N, [a_1 - a_2]_N, \ldots, [a_1 - a_k]_N)}{|x - a_1|^{n+s+1} \cdots |x - a_k|^{n+s+1}} dx \right| \leq$$
$$\frac{C z(\epsilon_1)\, \epsilon_1^{1-s}}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} \left(\prod_{j=2}^{k} |a_1 - a_j| \right) \left(\sum_{i=2}^{k} \prod_{\substack{j=2 \\ j\neq i}}^{k} (\epsilon_1 + |a_1 - a_j|) + (b_0\delta)^{k-1} \right).$$

The existence of the function G of the statement follows. $\square$

We are in a position to prove the nonlocal Piola Identity. Henceforth, supp denotes the support of a function.

Theorem 7.3.2. *Let $k \in \mathbb{N}$ be with $1 \leq k \leq n$. Consider indices $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$ and the functions*

$$[\cdot]_M = [\cdot]_{M_{i_1,\dots,i_k;j_1,\dots,j_k}}, \quad [\cdot]_{\bar{M}} = [\cdot]_{\bar{M}_{i_1,\dots,i_k;j_1,\dots,j_k}}$$

of Definition 4.1.1. Let $u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and $s \in (0,1)$. Then

$$\operatorname{Div}_\delta^s([\operatorname{cof}[D_\delta^s u]_M]_{\bar{M}}) = 0.$$

Proof. Let

$$[\cdot]_N = [\cdot]_{N_{j_1,\dots,j_k}}, \quad [\cdot]_{\bar{N}} = [\cdot]_{\bar{N}_{j_1,\dots,j_k}}$$

be the maps of Definition 4.1.1. Naturally, $\operatorname{Div}_\delta^s([\operatorname{cof}[D_\delta^s u]_M]_{\bar{M}}) = 0$ if and only if

$$\operatorname{div}_\delta^s[(\operatorname{cof}[D_\delta^s u]_M)_{i_\ell}]_{\bar{N}} = 0, \qquad \ell = 1, \dots, k.$$

We shall show $\operatorname{div}_\delta^s[(\operatorname{cof}[D_\delta^s u]_M)_{i_1}]_{\bar{N}} = 0$. The rest of the rows would proceed analogously.

Using (7.5), we have that, for a.e. $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\frac{(-1)^{k-1}}{c_{n,s}^k} \operatorname{div}_\delta^s[(\operatorname{cof}[D_\delta^s u]_M)_{i_1}]_{\bar{N}}(x) = \\
&\frac{(-1)^{k-1}}{c_{n,s}^{k-1}} \operatorname{pv}_x \int_{B(x,\delta)} \frac{[(\operatorname{cof}[D_\delta^s u]_M)_{i_1}]_{\bar{N}}(x')}{|x'-x|^{n+s+1}} \cdot (x'-x) w_\delta(x'-x)\, dx'.
\end{aligned} \tag{7.22}$$

Now, by (4.4) and (7.5), we have that for a.e. $x, x' \in \mathbb{R}^n$,

$$\begin{aligned}
&\frac{(-1)^{k-1}}{c_{n,s}^{k-1}} \frac{[(\operatorname{cof}[D_\delta^s u]_M)_{i_1}]_{\bar{N}}(x')}{|x'-x|^{n+s+1}} \cdot (x'-x) w_\delta(x'-x) \\
&= \frac{(-1)^{k-1}}{c_{n,s}^{k-1}} \frac{(\operatorname{cof}[D_\delta^s u]_M)_{i_1}(x')}{|x'-x|^{n+s+1}} \cdot [x'-x]_N w_\delta(x'-x) \\
&= \frac{(-1)^{k-1}}{c_{n,s}^{k-1}} \frac{\det\left([x'-x]_N, [D_\delta^s u_{i_2}(x')]_N, \dots, [D_\delta^s u_{i_k}(x')]_N\right)}{|x'-x|^{n+s+1}} w_\delta(x'-x) \\
&= \det\Bigg(\frac{[x'-x]_N w_\delta(x'-x)}{|x'-x|^{n+s+1}},\ \operatorname{pv}_{x'} \int \frac{u_{i_2}(y_2)[x'-y_2]_N}{|x'-y_2|^{n+s+1}} w_\delta(x'-y_2)\, dy_2, \dots, \\
&\qquad\qquad \operatorname{pv}_{x'} \int \frac{u_{i_k}(y_k)[x'-y_k]_N}{|x'-y_k|^{n+s+1}} w_\delta(x'-y_k)\, dy_k \Bigg) \\
&= \lim_{\varepsilon_2 \to 0} \cdots \lim_{\varepsilon_k \to 0} f^x_{\varepsilon_2,\dots,\varepsilon_k}(x'),
\end{aligned} \tag{7.23}$$

where for each $x \in \mathbb{R}^n$ and $\varepsilon_2, \dots, \varepsilon_k > 0$, we have defined $f^x_{\varepsilon_2,\dots,\varepsilon_k} : \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned}
&f^x_{\varepsilon_2,\dots,\varepsilon_k}(x') := \\
&\det\left(\frac{[x'-x]_N w_\delta(x'-x)}{|x'-x|^{n+s+1}}, \int_{B(x',\varepsilon_2)^c} \frac{u_{i_2}(y_2)[x'-y_2]_N}{|x'-y_2|^{n+s+1}} w_\delta(x'-y_2)\,dy_2, \dots, \right.\\
&\qquad\left. \int_{B(x',\varepsilon_k)^c} \frac{u_{i_k}(y_k)[x'-y_k]_N}{|x'-y_k|^{n+s+1}} w_\delta(x'-y_k)\,dy_k\right)
\end{aligned}$$

and we have used the continuity of the determinant. From now on, for the sake of clarity, since the compact support of the domain of most integrands is actually determined by that of w_δ we will avoid writing it in the integration domain and we will write $\int g = \int_{\mathbb{R}^n} g$.

By odd symmetry, we have that

$$\begin{aligned}
&\int_{B(x',\varepsilon_j)^c} u_{i_\ell}(y_\ell) \frac{[x'-y_\ell]_N}{|x'-y_\ell|^{n+s+1}} w_\delta(x'-y_\ell) dy_\ell = \\
&\int_{B(x',\delta)\setminus B(x',\varepsilon_j)} u_{i_\ell}(y_\ell) \frac{[x'-y_\ell]_N}{|x'-y_\ell|^{n+s+1}} w_\delta(x'-y_\ell) dy_\ell = \\
&\int_{B(x',\delta)\setminus B(x',\varepsilon_j)} \left(u_{i_\ell}(y_\ell) - u_{i_\ell}(x')\right) \frac{[x'-y_\ell]_N}{|x'-y_\ell|^{n+s+1}} w_\delta(x'-y_\ell) dy_\ell,
\end{aligned}$$

so, using the fact that u is Lipschitz and that w_δ is bounded by C_0, we have, for some constant $L > 0$, that

$$\begin{aligned}
&\left| \int_{B(x',\varepsilon_j)^c} u_{i_\ell}(y_\ell) \frac{[x'-y_\ell]_N}{|x'-y_\ell|^{n+s+1}} w_\delta(x-y_\ell) dy_\ell \right| \leq \\
&\int_{B(x',\delta)} \frac{C_0 \left|u_{i_\ell}(y_\ell) - u_{i_\ell}(x')\right|}{|x'-y_\ell|^{n+s}} dy_\ell \leq \\
&C_0 L \int_{B(x',\delta)} \frac{1}{|x'-y_\ell|^{n+s-1}} dy_\ell = C_0 L \int_{B(0,\delta)} \frac{1}{|y|^{n+s-1}} dy < \infty.
\end{aligned}$$

This shows that

$$\left|f^x_{\varepsilon_2,\dots,\varepsilon_k}(x')\right| \leq \frac{c}{|x'-x|^{n+s}}$$

for some $c > 0$ only depending on u and n. As

$$\int_{B(x,\varepsilon_1)^c} \frac{1}{|x'-x|^{n+s}} dx' < \infty,$$

for any $\varepsilon_1 > 0$, we can apply dominated convergence to conclude that

$$\int_{B(x,\varepsilon_1)^c} \lim_{\varepsilon_2\to 0} \cdots \lim_{\varepsilon_k \to 0} f^x_{\varepsilon_2,\dots,\varepsilon_k}(x')\,dx' = \lim_{\varepsilon_2\to 0} \cdots \lim_{\varepsilon_k \to 0} \int_{B(x,\varepsilon_1)^c} f^x_{\varepsilon_2,\dots,\varepsilon_k}(x')\,dx'.$$

Recalling (7.22) and (7.23), with this we obtain that

$$\frac{(-1)^{k-1}}{c^k_{n,s}} \operatorname{div}^s_\delta[(\operatorname{cof}[D^s_\delta u]_M)_{i_1}]_{\bar N}(x) = \lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0} \cdots \lim_{\varepsilon_k \to 0} \int_{B(x,\varepsilon_1)^c} f^x_{\varepsilon_2,\dots,\varepsilon_k}(x')\,dx'. \tag{7.24}$$

Now for every $\varepsilon_1,\dots,\varepsilon_k > 0$ we define $D_{\varepsilon_1,\dots,\varepsilon_k} := B(x,\varepsilon_1) \cup \bigcup_{j=2}^k B(y_j,\varepsilon_j)$ and have that, denoting $\tilde y = y_2,\dots,y_k$, $d\tilde y = dy_2,\dots,dy_k$ and

$$W^\delta_{\tilde y}(x',x) = w_\delta(x'-x)w_\delta(x'-y_2)\dots w_\delta(x'-y_k),$$

thanks to the multilinearity of the determinant,

$$\begin{aligned}
&\int_{B(x,\varepsilon_1)^c} f^x_{\varepsilon_2,\dots,\varepsilon_k}(x')\,dx' \\
&= \int_{B(x,\varepsilon_1)^c}\int_{B(x',\varepsilon_2)^c}\cdots\int_{B(x',\varepsilon_k)^c} \\
&\left(\frac{\det([x'-x]_N, u_{i_2}(y_2)[x'-y_2]_N,\dots,u_{i_k}(y_k)[x'-y_k]_N)}{|x'-x|^{n+s+1}|x'-y_2|^{n+s+1}\cdots|x'-y_k|^{n+s+1}} W^\delta_{\tilde y}(x',x)\right)\ d\tilde y\,dx' \\
&= \int u_{i_k}(y_k)\cdots\int u_{i_2}(y_2)\int_{D^c_{\varepsilon_1,\dots,\varepsilon_k}} \\
&\qquad\left(\frac{\det([x'-x]_N, [x'-y_2]_N,\dots,[x'-y_k]_N)}{|x'-x|^{n+s+1}|x'-y_2|^{n+s+1}\cdots|x'-y_k|^{n+s+1}} W^\delta_{\tilde y}(x',x)\right)\ dx'\,d\tilde y,
\end{aligned}$$

Set

$$\begin{aligned}
&g(x,x',y_2,\dots,y_k) := \\
&\frac{\det([x'-x]_N, [x'-y_2]_N,\dots,[x'-y_k]_N)}{|x'-x|^{n+s+1}|x'-y_2|^{n+s+1}\cdots|x'-y_k|^{n+s+1}} W^\delta_{\tilde y}(x',x).
\end{aligned}$$

Then,

$$\begin{aligned}
&\left|\int_{B(x,\varepsilon_1)^c} f^x_{\varepsilon_2,\dots,\varepsilon_k}(x')\,dx'\right| \le \\
&\|u\|^{k-1}_\infty \int_{\operatorname{supp} u}\cdots\int_{\operatorname{supp} u}\left|\int_{D^c_{\varepsilon_1,\dots,\varepsilon_k}} g(x,x',y_2,\dots,y_k)\,dx'\right| dy_2\cdots dy_k.
\end{aligned} \tag{7.25}$$

Thanks to Lemma 7.3.1,

$$\left| \int_{D^c_{\varepsilon_1,\ldots,\varepsilon_k}} g(x,x',y_2,\ldots,y_k)\,dx' \right| \leq \frac{\epsilon_k^{1-s}}{(\epsilon_1\cdots\epsilon_{k-1})^{n+s+2}} G(\epsilon_k, x-y_k, y_2-y_k,\ldots,y_{k-1}-y_k), \tag{7.26}$$

where G is the function that appears therein. Integrating in (7.26), we find that

$$\int_{\operatorname{supp} u} \cdots \int_{\operatorname{supp} u} \left| \int_{D^c_{\varepsilon_1,\ldots,\varepsilon_k}} g(x,x',y_2,\ldots,y_k)\,dx' \right| dy_2\cdots dy_k \leq h(\varepsilon_k,x)\,\frac{\epsilon_k^{1-s}}{(\epsilon_1\cdots\epsilon_{k-1})^{n+s+2}},$$

for some continuous functions $h:[0,\infty)\times\mathbb{R}^n\to[0,\infty)$ Consequently,

$$\lim_{\varepsilon_k\to 0}\int_{\operatorname{supp} u} \cdots \int_{\operatorname{supp} u} \left| \int_{D^c_{\varepsilon_1,\ldots,\varepsilon_k}} g(x,x',y_2,\ldots,y_k)\,dx' \right| dy_2\cdots dy_k = 0,$$

and, in view of (7.24) and (7.25), we obtain that $\operatorname{div}^s_\delta[(\operatorname{cof}[D^s_\delta u]_M)_{i_1}]_{\bar{N}}(x) = 0$. □

7.3.2 Weak continuity of $\det D^s_\delta u$

In this section we prove that any minor (determinant of a submatrix) of $D^s_\delta u$ is a weakly continuous mapping in $H^{s,p,\delta}_0(\Omega_\delta)$. We start by expressing a nonlocal integration by parts formula for the minors of $D^s_\delta u$ that involves the operator $K^{s,\delta}_\varphi$ of Lemma 6.2.2. Recall that for any $F\in\mathbb{R}^{n\times n}$ and $1\leq i\leq n$ we denote by F_i the i-th row of F.

Lemma 7.3.3. *Consider indices $1\leq i_1<\cdots<i_k\leq n$ and $1\leq j_1<\cdots<j_k\leq n$ and the functions*

$$[\cdot]_M=[\cdot]_{M_{i_1,\ldots,i_k;j_1,\ldots,j_k}},\quad [\cdot]_{\bar{M}}=[\cdot]_{\bar{M}_{i_1,\ldots,i_k;j_1,\ldots,j_k}},\quad [\cdot]_{\tilde{N}}=[\cdot]_{\tilde{N}_{i_1,\ldots,i_k}}$$

of Definition 4.1.1. Let $p\geq k-1$, $q\geq\frac{p}{p-1}$, $0<\delta$ and $0<s<1$. Let $u\in H^{s,p,\delta}_0(\Omega_{-\delta},\mathbb{R}^n)$ be such that $\operatorname{cof}[D^s_\delta u]_M\in L^q(\Omega,\mathbb{R}^{k\times k})$. Then, $\det[D^s_\delta u]_M\in L^1(\Omega)$, and for every $\varphi\in C^\infty_c(\Omega)$ we have that $[u]_{\tilde{N}}\cdot K^{s,\delta}_\varphi([\operatorname{cof}[D^s_\delta u]_M]_{\bar{M}})\in L^1(\Omega)$ and

$$\int_\Omega \det[D^s_\delta u]_M(x)\,\varphi(x)\,dx = -\frac{1}{k}\int_\Omega [u]_{\tilde{N}}(x)\cdot K^{s,\delta}_\varphi([\operatorname{cof}[D^s_\delta u]_M]_{\bar{M}})(x)\,dx \tag{7.27}$$

Proof. The fact $\det[D^s_\delta u]_M \in L^1(\Omega)$ is a consequence of formula (4.4) and Hölder's inequality, since $q \geq \frac{p}{p-1}$. Moreover, $[u]_{\tilde{N}} \cdot K^{s,\delta}_\varphi([\mathrm{cof}[D^s_\delta u]_M)]_{\bar{M}}) \in L^1(\Omega)$, since $[u]_{\tilde{N}} \in L^p(\Omega_\delta, \mathbb{R}^n)$ and

$$K^{s,\delta}_\varphi([\mathrm{cof}[D^s_\delta u]_M)]_{\bar{M}}) \in L^r(\Omega, \mathbb{R}^n)$$

for all $r \in [1, q]$ thanks to Lemma 6.2.2 .

Assume first $u \in C^\infty_c(\Omega_{-\delta}, \mathbb{R}^n)$ and let $\psi \in C^\infty_c(\Omega_{-\delta})$. Fix $x \in \mathbb{R}^n$ and $i \in \{i_1, \ldots, i_k\}$. Applying Lemma 6.2.5 and Theorem 7.3.2 to each row of $[\mathrm{cof}[D^s_\delta u]_M]_{\bar{M}}$,

$$\mathrm{div}^s_\delta\left(\psi\left([\mathrm{cof}[D^s_\delta u]_M]_{\bar{M}}\right)_i\right)(x) = K^{s,\delta}_\psi\left(([\mathrm{cof}[D^s_\delta u]_M]_{\bar{M}})^T_i\right)(x).$$

When we apply Theorem 6.1.2 to the constant function 1, we obtain from integration of the previous formula that

$$\begin{aligned}
0 = &\int_\Omega \mathrm{div}^s_\delta\left(\psi\left([\mathrm{cof}[D^s_\delta u]_M]_{\bar{M}}\right)_i\right)(x)\,dx + \\
&\int_\Omega \psi(x)[\mathrm{cof}[D^s_\delta u]_M]_{\bar{M}}(x) \cdot \int_{\Omega_{B,\delta}} \frac{x-y}{|x-y|^{n+s+1}} w_\delta(x-y))dydx \\
= &\int_\Omega K^{s,\delta}_\psi\left(([\mathrm{cof}[D^s_\delta u]_M]_{\bar{M}})^T_i\right)(x)\,dx + \\
&\int_\Omega \psi(x)[\mathrm{cof}[D^s_\delta u]_M]_{\bar{M}}(x) \cdot \int_{\Omega_{B,\delta}} \frac{x-y}{|x-y|^{n+s+1}} w_\delta(x-y))dydx.
\end{aligned} \tag{7.28}$$

By the definition of $K^{s,\delta}_\psi$ (Lemma 6.2.2),

$$\begin{aligned}
&\int_\Omega K^{s,\delta}_\psi\left(([\mathrm{cof}[D^s_\delta u]_M]_{\bar{M}})^T_i\right)(x)\,dx = \\
&c_{n,s}\int_\Omega\int_{\Omega_\delta} \frac{\psi(x)-\psi(y)}{|x-y|^{n+s}}(\mathrm{cof}[[D^s_\delta u]_M]_{\bar{M}})^T_i(y)\frac{y-x}{|y-x|}w_\delta(x-y)\,dydx
\end{aligned}$$

and that of the nonlocal gradient (Definition 6.1.1),

$$\begin{aligned}
&\int_\Omega D^s_\delta\psi(y) \cdot \left([\mathrm{cof}[D^s_\delta u]_M]_{\bar{M}}\right)_i(y)\,dy = \\
&c_{n,s}\int_\Omega\int_{\Omega_\delta} \frac{\psi(y)-\psi(x)}{|y-x|^{n+s}}(\mathrm{cof}[[D^s_\delta u]_M]_{\bar{M}})^T_i(y)\frac{y-x}{|y-x|}w_\delta(x-y)\,dxdy.
\end{aligned}$$

If we add now the remaining terms, we obtain an equality relating both integrals. Recalling the compact supports of ψ and w_δ, by Fubini's theorem we have

$$
\begin{aligned}
&\int_\Omega K_\psi^{s,\delta}\left(([\mathrm{cof}[D_\delta^s u]_M]_{\bar M})_i^T\right)(x)\,dx = \\
&\int D_\delta^s\psi(y)\cdot([\mathrm{cof}[D_\delta^s u]_M)]_{\bar M})_i\,(y)\,dy - \\
c_{n,s}&\int_\Omega\int_{\Omega_{B,\delta}}\frac{\psi(y)-\psi(x)}{|y-x|^{n+s}}(\mathrm{cof}[[D_\delta^s u]_M]_{\bar M})_i^T(y)\frac{y-x}{|y-x|}w_\delta(x-y)\,dxdy + \\
c_{n,s}&\int_\Omega\int_{\Omega_{B,\delta}}\frac{\psi(x)-\psi(y)}{|x-y|^{n+s}}(\mathrm{cof}[[D_\delta^s u]_M]_{\bar M})_i^T(y)\frac{y-x}{|y-x|}w_\delta(x-y)\,dydx.
\end{aligned}
$$

Since $\mathrm{supp}\, D_\delta^s u\subset\Omega$ and so is $\mathrm{supp}\,\psi$,

$$
\begin{aligned}
&\int_\Omega K_\psi^{s,\delta}\left(([\mathrm{cof}[D_\delta^s u]_M]_{\bar M})_i^T\right)(x)\,dx = \\
&\int D_\delta^s\psi(y)\cdot([\mathrm{cof}[D_\delta^s u]_M)]_{\bar M})_i\,(y)\,dy - \\
c_{n,s}&\int_\Omega \psi(y)(\mathrm{cof}[[D_\delta^s u]_M]_{\bar M})_i(y)\cdot\int_{\Omega_{B,\delta}}\frac{1}{|y-x|^{n+s}}\frac{y-x}{|y-x|}w_\delta(x-y)\,dxdy
\end{aligned}
$$

Thus, combining this with (7.28) we have the equality

$$\int_\Omega D_\delta^s\psi(y)\cdot([\mathrm{cof}[D_\delta^s u]_M)]_{\bar M})_i\,(y)\,dy = 0. \tag{7.29}$$

Now we assume that $u\in H_0^{s,p,\delta}(\Omega_{-\delta},\mathbb{R}^n)$ with $\mathrm{cof}[D_\delta^s u]_M\in L^q(\Omega,\mathbb{R}^{k\times k})$, and, again $\psi\in C_c^\infty(\Omega)$. By definition of $H_0^{s,p,\delta}(\Omega_{-\delta})$, let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $C_c^\infty(\Omega_{-\delta},\mathbb{R}^n)$ converging to u in $H^{s,p,\delta}(\Omega,\mathbb{R}^n)$. Then $[D_\delta^s u_j]_M\to[D_\delta^s u]_M$ in $L^p(\Omega,\mathbb{R}^{k\times k})$ and, hence, $\mathrm{cof}[D_\delta^s u_j]_M\to\mathrm{cof}[D_\delta^s u]_M$ in $L^{\frac{p}{k-1}}(\Omega,\mathbb{R}^{k\times k})$, so $[\mathrm{cof}[D_\delta^s u_j]_M]_{\bar M}\to[\mathrm{cof}[D_\delta^s u]_M]_{\bar M}$ in $L^{\frac{p}{k-1}}(\Omega,\mathbb{R}^{n\times n})$. Therefore, (7.29) holds as well, since $D_\delta^s\psi\in L^r(\Omega)$ for all $r\in[1,\infty]$ (see Lemma 6.2.1). Now let $\psi\in H_0^{s,p,\delta}(\Omega_{-\delta})$, and let $\{\psi_j\}_{j\in\mathbb{N}}$ be a sequence in $C_c^\infty(\Omega_{-\delta})$ converging to ψ in $H^{s,p,\delta}(\Omega)$. Then, $D_\delta^s\psi_j\to D_\delta^s\psi$ in $L^r(\Omega)$ for all $r\in[1,p]$. As $[\mathrm{cof}[D_\delta^s u]_M]_{\bar M}\in L^q(\Omega,\mathbb{R}^{n\times n})$, we have that (7.29) holds as well. To sum up, formula (7.29) is valid for any $u\in H_0^{s,p,\delta}(\Omega_{-\delta})$ with $\mathrm{cof}[D_\delta^s u]_M\in L^q(\Omega,\mathbb{R}^{k\times k})$ and any $\psi\in H_0^{s,p,\delta}(\Omega_{-\delta})$.

We apply (7.29) to $\psi=\varphi u_i$, which is in $H_0^{s,p,\delta}(\Omega_{-\delta})$ thanks to Lemma 6.2.3, and has compact support. By the formula for $D_\delta^s\psi$ given by Lemma

6.2.3, we obtain that

$$
\begin{aligned}
0 = &\int_\Omega \varphi(y)\, D^s_\delta u_i(y) \cdot \left([\operatorname{cof}[D^s_\delta u]_M)]_{\bar{M}} \right)_i (y)\, dy + \\
&\int_\Omega K^{s,\delta}_\varphi (u_i I)(y) \cdot \left([\operatorname{cof}[D^s_\delta u]_M)]_{\bar{M}} \right)_i (y)\, dy.
\end{aligned}
\tag{7.30}
$$

Using formula (4.4), the fact $i \in \{i_1, \ldots, i_k\}$ and elementary properties of the functions of Definition 4.1.1, we find that for any $F \in \mathbb{R}^{n\times n}$,

$$
F_i \cdot \left([\operatorname{cof}[F]_M]_{\bar{M}} \right)_i = \det[F]_M .
$$

Since $\operatorname{supp} u_i \subset \Omega_{-\delta}$, the integral defining $K^{s,\delta}_\varphi(u_i I)$ has Ω as integration domain. Then, using this and Fubini's theorem, from (4.28) we arrive at

$$
\begin{aligned}
0 = &\int_\Omega \varphi(y) \det[D^s_\delta u]_M(y)\, dy\; + \\
&c_{n,s} \int_\Omega u_i(x) \int_\Omega \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+s}} \left([\operatorname{cof}[D^s_\delta u]_M)]_{\bar{M}} \right)_i (y) \cdot \frac{x-y}{|x-y|}\, dy\, dx.
\end{aligned}
$$

We sum this equality for $i = i_1, \ldots, i_k$ and obtain that

$$
\begin{aligned}
0 = &k \int_\Omega \varphi(y) \det[D^s_\delta u]_M(y)\, dy\; + \\
&c_{n,s} \int_\Omega [u]_{\bar{N}}(x) \cdot \int_\Omega \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+s}} \left([\operatorname{cof}[D^s_\delta u]_M)]_{\bar{M}} \right) (y) \frac{x-y}{|x-y|}\, dy\, dx,
\end{aligned}
$$

which, recalling again that $\operatorname{supp}[\operatorname{cof}[D^s_\delta u]_M)]_{\bar{M}} \subset \Omega$, is the required formula. □

Now we establish the closedness and continuity properties of the minors of $D^s_\delta u$ in the weak topology of $H^{s,p,\delta}$. In the notation of Definition 4.1.1 *a)*, a minor of order k is a function $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ such that there exist $1 \le i_1 < \cdots < i_k \le n$ and $1 \le j_1 < \cdots < j_k \le n$ for which $\mu(F) = \det[F]_M$ for all $F \in \mathbb{R}^{n\times n}$. Recall the notation p^*_s of Theorem 6.4.5, and the affine space $H^{s,p}_g$.

Theorem 7.3.4. *Let $p \ge n-1$ and $0 < s < 1$. Let $g \in H^{s,p,\delta}(\Omega)$ and $u \in H^{s,p,\delta}_g(\Omega_{-\delta}, \mathbb{R}^n)$. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $H^{s,p,\delta}_g(\Omega_{-\delta}, \mathbb{R}^n)$ such that $u_j \rightharpoonup u$ in $H^{s,p,\delta}(\Omega, \mathbb{R}^n)$. Then*

a) If $k \in \mathbb{N}$ with $1 \le k \le n-2$ and μ is a minor of order k then $\mu(D^s_\delta u_j) \rightharpoonup \mu(D^s_\delta u)$ in $L^{\frac{p}{k}}(\Omega)$ as $j \to \infty$.

b) If $\operatorname{cof} D^s_\delta u_j \rightharpoonup \vartheta$ *in* $L^q(\Omega, \mathbb{R}^{n\times n})$ *for some* $q \in [1,\infty)$ *and* $\vartheta \in L^q(\Omega, \mathbb{R}^{n\times n})$ *then* $\vartheta = \operatorname{cof} D^s_\delta u$.

c) Assume $\det D^s_\delta u_j \rightharpoonup \theta$ *in* $L^\ell(\mathbb{R}^n)$ *for some* $\ell \in [1,\infty)$ *and some* $\theta \in L^\ell(\mathbb{R}^n)$. *If* $sp < n$ *assume, in addition, that* $\operatorname{cof} D^s_\delta u_j \rightharpoonup \operatorname{cof} D^s_\delta u$ *in* $L^q(\mathbb{R}^n, \mathbb{R}^{n\times n})$ *for some* $q \in (\frac{p^*}{p^*-1}, \infty)$. *Then* $\theta = \det D^s_\delta u$.

Proof. Without loss of generality we assume $g = 0$, therefore, we can take $u \in H^{s,p,\delta}_0(\Omega_{-\delta})$.

We will prove *a)* by induction on k. For $k = 1$ the result is trivial. Assume it holds for some $k \leq n-3$ and let us prove it for $k+1$. Let μ be a minor of order $k+1$. In the notation of Definition 4.1.1 *a)*, $\mu(F) = \det[F]_M$ for all $F \in \mathbb{R}^{n\times n}$, where $[\cdot]_M = [\cdot]_{M_{i_1,\ldots,i_{k+1};j_1,\ldots,j_{k+1}}}$ for some $1 \leq i_1 < \cdots < i_{k+1} \leq n$ and $1 \leq j_1 < \cdots < j_{k+1} \leq n$. Let $\varphi \in C^\infty_c(\Omega)$. By induction assumption, $\operatorname{cof}[D^s_\delta u_j]_M \rightharpoonup \operatorname{cof}[D^s_\delta u]_M$ in $L^{\frac{p}{k}}(\Omega, \mathbb{R}^{(k+1)\times(k+1)})$ as $j \to \infty$, so $[\operatorname{cof}[D^s_\delta u_j]_M]_{\bar{M}} \rightharpoonup [\operatorname{cof}[D^s_\delta u]_M)]_{\bar{M}}$ in $L^{\frac{p}{k}}(\Omega, \mathbb{R}^{n\times n})$. By Lemma 6.2.2 , $K^{s,\delta}_\varphi([\operatorname{cof}[D^s_\delta u_j]_M]_{\bar{M}}) \rightharpoonup K^{s,\delta}_\varphi([\operatorname{cof}[D^s_\delta u]_M)]_{\bar{M}})$ in $L^r(\Omega, \mathbb{R}^n)$ for every $r \in [1, \frac{p}{k}]$. By Theorem 6.4.5, $[u_j]_{\tilde{N}} \to [u]_{\tilde{N}}$ in $L^p(\Omega)$, so

$$[u_j]_{\tilde{N}} \cdot K^{s,\delta}_\varphi([\operatorname{cof}[D^s_\delta u_j]_M]_{\bar{M}}) \rightharpoonup [u]_{\tilde{N}} \cdot K^{s,\delta}_\varphi([\operatorname{cof}[D^s_\delta u]_M)]_{\bar{M}}) \quad \text{in } L^1(\Omega) \tag{7.31}$$

since $\frac{k}{p} + \frac{1}{p} \leq 1$. We apply Lemma 7.3.3 and, in particular, formula (7.27) to conclude that

$$\int_\Omega \det[D^s_\delta u_j(x)]_M \, \varphi(x)\, dx \to \int_\Omega \det[D^s_\delta u(x)]_M \, \varphi(x)\, dx. \tag{7.32}$$

This shows that $\det[D^s_\delta u_j]_M \rightharpoonup \det[D^s_\delta u]_M$ in the sense of distributions. As $\{\det[D^s_\delta u_j]_M\}_{j\in\mathbb{N}}$ is bounded in $L^{\frac{p}{k+1}}(\Omega)$ and $p > k+1$, we have that $\det[D^s_\delta u_j]_M \rightharpoonup \det[D^s_\delta u]_M$ in $L^{\frac{p}{k+1}}(\Omega)$.

The proof of *b)* follows the lines of *a)*. Let μ be a minor of order $n-1$. In the notation of Definition 4.1.1 *a)*, $\mu(F) = \det[F]_M$ for all $F \in \mathbb{R}^{n\times n}$, where $[\cdot]_M = [\cdot]_{M_{i_1,\ldots,i_{n-1};j_1,\ldots,j_{n-1}}}$ for some $1 \leq i_1 < \cdots < i_{n-1} \leq n$ and $1 \leq j_1 < \cdots < j_{n-1} \leq n$. Let $\varphi \in C^\infty_c(\Omega)$. By part *a)*, $\operatorname{cof}[D^s_\delta u_j]_M \rightharpoonup \operatorname{cof}[D^s_\delta u]_M$ in $L^{\frac{p}{n-2}}(\Omega, \mathbb{R}^{(n-1)\times(n-1)})$, so $[\operatorname{cof}[D^s_\delta u_j]_M]_{\bar{M}} \rightharpoonup [\operatorname{cof}[D^s_\delta u]_M)]_{\bar{M}}$ in $L^{\frac{p}{n-2}}(\Omega, \mathbb{R}^{n\times n})$. We have that $K^{s,\delta}_\varphi([\operatorname{cof}[D^s_\delta u_j]_M]_{\bar{M}}) \rightharpoonup K^{s,\delta}_\varphi([\operatorname{cof}[D^s_\delta u]_M)]_{\bar{M}})$ in $L^r(\Omega, \mathbb{R}^n)$ for every $r \in [1, \frac{p}{n-2}]$, by Lemma 6.2.2. By Theorem 6.4.5, $[u_j]_{\tilde{N}} \to [u]_{\tilde{N}}$ in $L^p(\Omega)$, so convergence (7.31) is also valid since $\frac{n-2}{p} + \frac{1}{p} \leq 1$. Thanks to (7.27), we conclude that convergence (7.32) holds. This shows that $\mu(D^s_\delta u_j) \rightharpoonup \mu(D^s_\delta u)$ in the sense of distributions. As this is true for every minor μ of order $n-1$, we obtain that $\operatorname{cof} D^s_\delta u_j \rightharpoonup \operatorname{cof} D^s_\delta u$ in the sense of distributions. Thanks to the assumption, $\vartheta = \operatorname{cof} D^s_\delta u$.

We finally show part *c)*. Let $\varphi \in C^\infty_c(\Omega)$. Assume first $sp < n$. By the assumption and Lemma 6.2.2 , $K^{s,\delta}_\varphi(\operatorname{cof} D^s_\delta u_j) \rightharpoonup K^{s,\delta}_\varphi(\operatorname{cof} D^s_\delta u)$ in $L^r(\Omega, \mathbb{R}^n)$ for every $r \in [1, q]$. By Theorem 6.4.5, $u_j \to u$ in $L^t(\Omega)$ for every $t \in [1, p^*)$, so

$$u_j \cdot K^{s,\delta}_\varphi(\operatorname{cof} D^s_\delta u_j) \rightharpoonup u_j \cdot K^{s,\delta}_\varphi(\operatorname{cof} D^s_\delta u) \quad \text{in } L^1(\Omega) \tag{7.33}$$

since $\frac{1}{q} + \frac{1}{p^*_s} < 1$.

Assume now $sp \geq n$. Then $\{\operatorname{cof} D^s_\delta u_j\}_{j\in\mathbb{N}}$ is bounded in $L^{\frac{p}{n-1}}(\Omega, \mathbb{R}^{n\times n})$ so, thanks to part *b)*, $\operatorname{cof} D^s_\delta u_j \rightharpoonup \operatorname{cof} D^s_\delta u$ in $L^{\frac{p}{n-1}}(\Omega, \mathbb{R}^{n\times n})$. By Lemma 6.2.2 , $K^{s,\delta}_\varphi(\operatorname{cof} D^s_\delta u_j) \rightharpoonup K^{s,\delta}_\varphi(\operatorname{cof} D^s_\delta u)$ in $L^r(\Omega, \mathbb{R}^n)$ for every $r \in [1, \frac{p}{n-1}]$. By Theorem 6.4.5, $u_j \to u$ in $L^t(\Omega)$ for every $t \in [1, \infty)$, so convergence (4.31) holds since $p > n-1$.

In either case, we have convergence (7.33), so by (7.27) we obtain

$$\int_\Omega \det D^s_\delta u_j(x)\, \varphi(x)\, dx \to \int_\Omega \det D^s_\delta u(x)\, \varphi(x)\, dx.$$

This shows that $\det D^s_\delta u_j \rightharpoonup \det D^s_\delta u$ in the sense of distributions, so $\theta = \det D^s_\delta u$. $\square$

7.3.3 Existence of minimizers

In this section we prove the existence of minimizers in $H^{s,p,\delta}$ of functionals of the form

$$I(u) := \int_\Omega W(x, u(x), D^s_\delta u(x))\, dx. \tag{7.34}$$

under natural coercivity and polyconvexity (Definition 4.0.1) assumptions. +

The existence theorem of this chapter is as follows. Its proof relies on a standard argument in the calculus of variations, once we have the continuity (with respect to the weak convergence) of the minors given by Theorem 7.3.4.

Theorem 7.3.5. *Let $p \geq n-1$ satisfy $p > 1$, $\delta > 0$ and $0 < s < 1$. Let $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ satisfy the following conditions:*

a) W is $\mathcal{L}^n \times \mathcal{B}^n \times \mathcal{B}^{n\times n}$-measurable, where $\mathcal{L}^n$ denotes the Lebesgue sigma-algebra in $\mathbb{R}^n$, whereas $\mathcal{B}^n$ and $\mathcal{B}^{n\times n}$ denote the Borel sigma-algebras in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, respectively.

b) $W(x, \cdot, \cdot)$ is lower semicontinuous for a.e. $x \in \mathbb{R}^n$.

c) For a.e. $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}^n$, the function $W(x, y, \cdot)$ is polyconvex.

d) There exist a constant $c > 0$, an $a \in L^1(\mathbb{R}^n)$ and a Borel function $h : [0, \infty) \to [0, \infty)$ such that

$$\lim_{t\to\infty} \frac{h(t)}{t} = \infty$$

and, for some $q > \frac{p_s^*}{p_s^*-1}$ if $sp < n$,

$$\begin{cases} W(x, y, F) \geq a(x) + c\,|F|^p + c\,|\operatorname{cof} F|^q + h(|\det F|), & \text{if } sp < n, \\ W(x, y, F) \geq a(x) + c\,|F|^p\,, & \text{if } sp \geq n, \end{cases}$$

for a.e. $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$ and all $F \in \mathbb{R}^{n\times n}$.

Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $u_0 \in H^{s,p,\delta}(\Omega, \mathbb{R}^n)$. Define I as in (7.34), and assume that I is not identically infinity in $H^{s,p,\delta}_{u_0}(\Omega_{-\delta}, \mathbb{R}^n)$. Then there exists a minimizer of I in $H^{s,p,\delta}_{u_0}(, \Omega_{-\delta}\mathbb{R}^n)$.

Proof. Assumption *d)* shows that the functional I is bounded below by $\int a$. As I is not identically infinity in $H^{s,p,\delta}_{u_0}(\Omega_{-\delta}, \mathbb{R}^n)$, there exists a minimizing sequence $\{u_j\}_{j\in\mathbb{N}}$ of I in $H^{s,p,\delta}_{u_0}(\Omega_{-\delta}, \mathbb{R}^n)$. Assumption *d)* implies that $\{D^s_\delta u_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega, \mathbb{R}^{n\times n})$. We claim that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega_{-\delta}, \mathbb{R}^{n\times n})$. Indeed, in the case $sp < n$ we use Theorem 6.4.1; in the case $sp = n$ we use Theorem 6.4.1 and the embedding $H^{s,\frac{n}{s},\delta}(\Omega) \subset H^{s,q,\delta}(\Omega)$ for all $q < \frac{n}{s}$. Finally, in the case $sp > n$ we use the embedding provided by Theorem 6.4.3. Thus, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega_{-\delta}, \mathbb{R}^{n\times n})$. As $u_j = u_0$ in $\Omega_\delta \backslash \Omega_{-\delta}$ for all $j \in \mathbb{N}$, we also have that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega_\delta, \mathbb{R}^n)$, and, consequently, also in $H^{s,p,\delta}(\Omega, \mathbb{R}^n)$. As $H^{s,p,\delta}(\Omega, \mathbb{R}^n)$ is reflexive, we can extract a weakly convergent subsequence. Using Theorem 6.4.5, we obtain that there exists $u \in H^{s,p,\delta}_{u_0}(\Omega_{-\delta}, \mathbb{R}^n)$ such that for a subsequence (not relabelled),

$$u_j \rightharpoonup u \text{ in } H^{s,p,\delta}(\Omega, \mathbb{R}^n) \quad \text{and} \quad u_j \to u \text{ in } L^p(\Omega, \mathbb{R}^n). \tag{7.35}$$

Now, by Theorem 7.3.4, for any minor μ of order $k \leq n-2$, we have that

$$\mu(D^s_\delta u_j) \rightharpoonup \mu(D^s_\delta u) \text{ in } L^{\frac{p}{k}}(\Omega). \tag{7.36}$$

If $sp < n$ then, by assumption *d)*, $\{\operatorname{cof} D^s_\delta u_j\}_{j\in\mathbb{N}}$ is bounded in $L^q(\Omega, \mathbb{R}^{n\times n})$, whereas if $sp \geq n$ we call $q := \frac{p}{n-1}$ and have that $\{\operatorname{cof} D^s_\delta u_j\}_{j\in\mathbb{N}}$ is bounded in $L^q(\Omega, \mathbb{R}^{n\times n})$. In either case we have that $q > 1$, so for a subsequence $\{\operatorname{cof} D^s_\delta u_j\}_{j\in\mathbb{N}}$ converges weakly in $L^q(\Omega, \mathbb{R}^{n\times n})$ and, by Theorem 7.3.4,

$$\operatorname{cof} D^s_\delta u_j \rightharpoonup \operatorname{cof} D^s_\delta u \text{ in } L^q(\Omega, \mathbb{R}^{n\times n}). \tag{7.37}$$

If $sp < n$ then, by assumption *d)* and de la Vallée Poussin's criterion, $\{\det D^s_\delta u_j\}_{j\in\mathbb{N}}$ is equiintegrable, whereas if $sp \geq n$ we have that $\{\det D^s_\delta u_j\}_{j\in\mathbb{N}}$ is bounded in $L^{\frac{p}{n}}(\Omega)$ and $\frac{p}{n} > 1$. In either case we have that, for a subsequence $\{\det D^s_\delta u_j\}_{j\in\mathbb{N}}$ converges weakly in $L^\ell(\Omega)$ with

$$\begin{cases} \ell = 1 & \text{if } sp < n, \\ \ell = \frac{p}{n} & \text{if } sp \geq n, \end{cases}$$

and, hence, by Theorem 7.3.4,

$$\det D^s_\delta u_j \rightharpoonup \det D^s_\delta u \text{ in } L^\ell(\Omega). \tag{7.38}$$

Convergences (7.35)–(7.38) imply, thanks to a standard lower semicontinuity result for polyconvex functionals (see, e.g., [15, Th. 5.4] or [59, Th. 7.5]), that for any $R > 0$,

$$\int_\Omega W(x, u(x), D^s_\delta u(x))\, dx \leq \liminf_{j\to\infty} \int_\Omega W(x, u_j(x), D^s_\delta u_j(x))\, dx.$$

Therefore,

$$I(u) \leq \liminf_{j\to\infty} I(u_j).$$

Hence, u is a minimizer of I in $H^{s,p,\delta}_{u_0}(\Omega_{-\delta}, \mathbb{R}^n)$ and the proof is concluded. $\square$

Notice that the same comments after the existence (of minimizer) theorem in Section 4.3 regarding the examples from Section 3.6 are also valid in this case. In other words, this theorem determining the existence of minimizers of a nonlocal hyperelastic energy is compatible with functions exhibiting singularities of *fracture* and *cavitation* type.

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